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Thermal Buckling and Postbuckling of Euler–Bernoulli Beams Supported on Nonlinear Elastic Foundations

S.-R. Li*
Lanzhou University of Technology, 730050 Lanzhou, Gansu, People's Republic of China
and
R. C. Batra†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24060

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Deformations of homogeneous and isotropic pinned–pinned and fixed–fixed Euler–Bernoulli beams supported on nonlinear elastic foundations and heated uniformly into the postbuckling regime have been analyzed numerically. Geometric nonlinearities introduced by finite deflections and curvature of the deformed beams are incorporated in the problem formulation. First buckling due to the uniform temperature rise and buckling mode transitions are investigated analytically by analyzing the linear problem. Subsequently, the nonlinear boundary-value problems for postbuckling of beams are transformed into initial-value problems and analyzed by the shooting method. For different values of the elastic foundation parameters, postbuckled configurations are illustrated.

Nomenclature

A	=	area of the cross section of the beam
E	=	Young's modulus
f	=	dimensionless deflection of the beam center
H	=	resultant horizontal force
I	=	moment of inertia of the cross section
K_1, K_2	=	dimensionless values of k_1, k_2
$(K_1)_{mn}$	=	stiffness of elastic foundation at which buckling mode transitions
k_1, k_2	=	linear and cubic stiffness parameters of the elastic foundation
l	=	initial length of the beam
M	=	bending moment
m	=	dimensionless value of M
m_0	=	dimensionless bending moment at a fixed end
N	=	resultant axial force
P_H, P_V	=	dimensionless values of H and V
q_x, q_y	=	distributed mechanical loads on the beam
S	=	dimensionless value of s
s	=	arc length of the deformed centroidal axis
T	=	temperature rise
U	=	dimensionless value of u
u	=	displacement in x direction
V	=	resultant vertical force
W	=	dimensionless value of w
w	=	displacement in y direction or beam deflection
x, y	=	rectangular Cartesian coordinate axes
α	=	coefficient of thermal expansion
θ	=	angle between the deformed beam's axis and the x axis
θ_0	=	rotational angle of the pinned end of the beam
κ	=	curvature of the deformed beam axis
Λ	=	stretch of the centroidal axis
λ	=	slenderness ratio of the beam
ξ	=	dimensionless value of x
τ	=	dimensionless value of T

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*Professor, Department of Engineering Mechanics; lisr@lut.cn.
†Clifton C. Garvin Professor, Department of Engineering Science and Mechanics, M/C 0219; rbatra@vt.edu.

I. Introduction

It is well known that a beam heated from the stress free reference configuration with its edges constrained from moving axially will have an axial compressive stress developed in it. When this axial compressive stress reaches a critical value, the beam will buckle, and will go into a postbuckled configuration upon further heating. The postbuckling deformations of slender components, such as robotic arms, optical fibers, satellite tethers, microresonators, and microelectromechanical systems, are of interest for design purposes. Satellite tethers are subjected to large variations in temperature during their life cycle. Microelectromechanical resonators are fabricated as clamped–clamped composite beams, and they sometimes buckle during the manufacturing process. These are often used as filters and are important for mobile communication systems and signal processing applications. Also, the temperature rise caused by the heat produced by the electric current may be enough to induce buckling or change the buckling mode as discussed below.

The analysis of postbuckling behavior under thermal loads of railroad tracks, concrete pavements, and pipelines either buried underground or resting on ground is needed to fully comprehend their failure. These structural elements are exposed to different environmental conditions and hence to different thermal cycles. Here we consider a problem in the second category, and simplify it considerably by studying the buckling and postbuckling response of a uniformly heated Euler–Bernoulli beam resting on an elastic foundation. Furthermore, the elastic foundation is assumed to exert distributed forces on the beam; these forces may vary either linearly or nonlinearly with both the axial and the transverse displacements of a point. Even such a simple model involves nonlinear governing equations and illuminates interesting transitions among buckling modes. The mathematical model applies to a beam/rod wrapped in a thick rubberlike sleeve with the action of the sleeve replaced by distributed forces on the beam, and to a microelectromechanical resonator with the action of the electrode replaced by a distributed force on the substrate.

Emam [1] has reviewed the literature on vibrations of postbuckled Euler–Bernoulli beams, and has compared analytically predicted interactions among different modes with the experimental findings; the reader is referred to Emam's dissertation for historical developments in the field. Lestari and Hanagud [2] have studied the nonlinear vibrations of buckled beams with elastic end constraints, and the beam subjected simultaneously to axial and lateral loads. Other works [3–5] have analyzed either analytically or numerically nonlinear ordinary differential equations associated with the postbuckling behavior of initially straight rods subjected to compressive loads and different boundary conditions. Kreider and Nayfeh [6] have studied experimentally and theoretically the nonlinear vibrations of a clamped–clamped buckled beam. Phungpaingam et al. [7] have investigated the postbuckling response of a simple beam under an intermediate follower force acting along the tangent to the deformed axis of the beam. They found that the analytical solution obtained by solving the elliptic integral equations matched well with the numerical solution computed with the shooting method. Plaut [8] studied the postbuckling deformations and vibrations of end-supported elastica pipes carrying fluid and also subjected to follower loads. He employed the shooting method to solve the nonlinear boundary-value problem for the equilibrium configuration, and the linear boundary-value problem for frequencies of the first four modes of vibration.

Postbuckling deformations of perfect and geometrically imperfect elastic columns resting on an elastic Winkler foundation have been analyzed by Kounadis et al. [9]. They found that the critical state of a perfect column is a stable symmetric bifurcation point. Abu-Salih and Elata [10] assumed the beam to be infinite, perfectly bonded to a linear elastic foundation, and subjected to internal compressive stresses. They incorporated extensional deformations into the problem formulation, and found that the wave length of a postbuckled beam is unaffected by the magnitude of the internal stress. Li and Balachandran [11] have analyzed the buckling and free vibrations of composite microelectromechanical resonators modeled as a stepped composite Euler–Bernoulli beam. They accounted for axial deformations of the beam. It appears that postbuckling deformations of a thermally loaded Euler–Bernoulli beam resting on an elastic foundation have not been analyzed thus far.

A difference between the buckling of a beam due to the applied axial force and that due to temperature rise is that in the latter case axial deformations may not be negligible and ought to be considered. Jekot [12] has studied deformations of a thermally postbuckled beam made of a nonlinear thermoelastic material. However, the geometric nonlinearity introduced by the curvature of the centroidal axis of the deformed beam was not considered, and a simple expression for the axial strain was employed. Raju and Rao [13–18] have used the Rayleigh–Ritz and the finite element methods to analyze the thermal postbuckling of uniform and tapered columns supported on elastic foundations. They incorporated the geometric nonlinearity similar to that in the von Karman plate theory, but ignored the nonlinearity introduced by the curvature of the deformed centroidal axis.

For a pinned–pinned beam, Coffin and Bloom [19] accounted for the curvature of the beam deformed due to heating, or the absorption of water, and formulated the problem for the postbuckled beam in terms of two coupled elliptic integral equations that were solved numerically. For a nonlinear thermal strain-temperature relation, Vaz and Solano [20,21] investigated postbuckling deformations of pinned–pinned rods and gave a closed-form solution via uncoupled elliptic integrals. However, their analysis is valid only for pinned–pinned boundary conditions. Based on the geometric nonlinear theory of extensible beams and with temperature increasing uniformly throughout the beam, Li and Cheng [22] and Li et al. [23] have developed mathematical models of postbuckled Euler–Bernoulli beams with pinned–pinned, fixed–fixed, and pinned–fixed boundary conditions. The resulting nonlinear boundary-value problems were solved numerically with the shooting method in conjunction with concepts of analytical continuation. The work has been extended [24] to bending and buckling of nonlinear Timoshenko beams subjected to thermomechanical loads. Wu and Zhong [25] studied postbuckling and imperfection sensitivity of axially compressed beams with clamped–clamped and fixed–free ends resting on an elastic Winkler foundation. They [25] found that for a linear Winkler foundation and a large range of values of the foundation stiffness parameter, a measure of the beam deformation increases with a decrease in the load during its postbuckling deformations, and concluded that the postbuckled paths are unstable.

Here we consider geometric nonlinearities, and develop a mathematical model of thermal buckling and postbuckling of an elastic Euler–Bernoulli beam resting on a nonlinear elastic foundation. The load applied by the elastic foundation is assumed to depend upon two components of displacement of a point on the centroidal axis of the beam. For a satellite tether, the force exerted by the environment could conceivably be modeled by a similar relation. For a microelectromechanical resonator, the force exerted by the piezoelectric layer on the substrate can be approximated in a similar way. Because deformations of a hygroscopic body due to moisture absorption are similar to those of a thermoelastic body due to temperature rise, the analysis also applies to beams immersed in water with the action of water on the beam approximated by distributed forces depending upon the displacements of a point on the centroidal axis of the beam. The nonlinear boundary-value problem is numerically solved with the shooting method for postbuckled configurations of beams and their equilibrium paths. Computed results are compared with those available in the literature. We delineate effects of parameters of the elastic foundation on the beam's deformations, and give values of the foundation stiffness at which the transition in buckling modes occurs.

II. Mathematical Model

We consider a uniform elastic beam of initial length l resting on a nonlinear elastic foundation and having its ends constrained from moving axially; a schematic sketch of the problem studied is shown in Fig. 1. An axially distributed mechanical load $\boldsymbol{q} = (q_x, q_y)$ and a uniform slow T deform the beam from its natural state. Assume that E and α are independent of T, and a plane section initially perpendicular to the centroidal axis of the beam remains plane and becomes perpendicular to the centroidal axis of the deformed beam. We denote a point of the centroidal axis of the undeformed beam as C: (x, y) with $x \in [0, l]$, $y \equiv 0$, in which x and y are coordinates of a point in a rectangular Cartesian coordinate system. When the beam is deformed, as shown in Fig. 2, the material point C moves to the point C': $(X, Y) = [x + u(x), w(x)]$, in which $u(x)$ and $w(x)$ are displacements of the point C in the x and the y directions, respectively. Here we have presumed that the centroidal axis of the deformed beam is in the xy plane. Thus,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \Lambda, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \Lambda \cos\theta - 1, \qquad \frac{\mathrm{d}w}{\mathrm{d}x} = \Lambda \sin\theta \tag{1}$$

where

$$\Lambda = \sqrt{(1 + \mathrm{d}u/\mathrm{d}x)^2 + (\mathrm{d}w/\mathrm{d}x)^2} \tag{2}$$

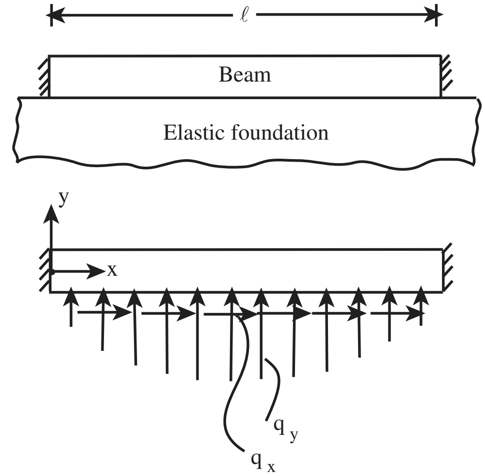


Fig. 1 (Top) A schematic sketch of the problem studied, and (bottom) a representation of forces exerted by the elastic foundation on the beam.

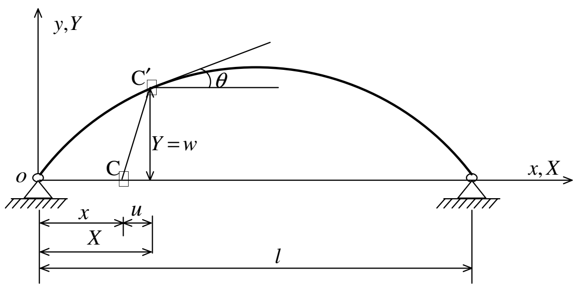


Fig. 2 A buckled configuration of a pinned–pinned beam.

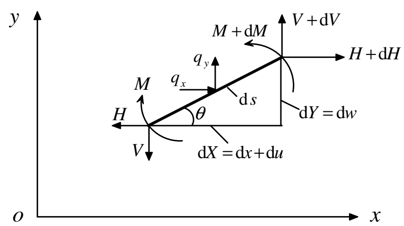


Fig. 3 Free-body diagram of a differential element.

A free-body diagram of a differential element of the beam is exhibited in Fig. 3. The equilibrium of forces and of moments acting on a segment, ds, of the deformed beam yield

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\Lambda q_x, \qquad \frac{\mathrm{d}V}{\mathrm{d}x} = -\Lambda q_y, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = \Lambda(H\sin\theta - V\cos\theta) \tag{3}$$

For a beam resting on a nonlinear elastic foundation, the force exerted by the foundation on the beam opposes displacements of the beam's centroidal axis, and can be expressed as

$$\boldsymbol{q} = -(k_1 + k_2\delta^2)\boldsymbol{\delta} \tag{4}$$

in which $\boldsymbol{\delta} = \{u, w\}^T$ is the displacement vector of a point of the centroidal axis, and

$$u = \delta\cos\theta, \qquad w = \delta\sin\theta \tag{5}$$

Because $\delta = \sqrt{u^2 + w^2}$, therefore

$$q_x = -k_1 u - k_2 u(u^2 + w^2), \qquad q_y = -k_1 w - k_2 w(u^2 + w^2) \tag{6}$$

When the resistance offered by the elastic foundation due to the axial displacement u can be neglected, then Eq. (6) is simplified to

$$q_x = 0, \qquad q_y = -k_1 w - k_2 w^3 \tag{7}$$

For an isotropic and homogeneous elastic beam resting on an isotropic and homogeneous elastic foundation, assumption (4) for the force exerted by the foundation on the beam is reasonable. One could, in principle, consider different values of stiffness parameters k_1 and k_2 for forces in the x and the y directions, respectively.

Constitutive relations for the thermoelastic beam are taken to be

$$N = EA(\Lambda - 1 - \alpha T) \tag{8a}$$

$$M = -\frac{EI}{\Lambda}\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{8b}$$

in which EA and EI are, respectively, the same as the extensional and the bending rigidities of a linear elastic beam. Furthermore, T is assumed to be uniform throughout the beam, and N is perpendicular to the cross section. We note that Eqs. (8a) and (8b) are not linear in displacement gradients. N is related to H and V by

$$N = H\cos\theta + V\sin\theta \tag{9}$$

Substituting for N from Eq. (9) into Eq. (8a), and solving the resulting equation for Λ we get

$$\Lambda = [(H\cos\theta + V\sin\theta)/EA] + \alpha T + 1 \tag{10}$$

Equation (8b), and equations obtained by substituting from Eqs. (6) and (10) into Eqs. (1) and (3), form seven coupled nonlinear differential equations for the seven unknown functions $s(x)$, $u(x)$, $w(x)$, $\theta(x)$, $H(x)$, $V(x)$, and $M(x)$ defined on the interval [0, l].

By introducing the following nondimensional quantities

$$(\xi, S, U, W) = (x, s, u, w)/l, \qquad \lambda = l(A/I)^{1/2}$$
$$(K_1, K_2) = (k_1, k_2)l^4/EI \tag{11a}$$

$$\tau = \alpha\lambda^2 T, \qquad (P_H, P_V) = l^2(H, V)/EI, \qquad m = lM/EI \tag{11b}$$

and substituting them into Eqs. (1), (3), (8), and (10), we arrive at the following equations in terms of nondimensional variables:

$$\frac{\mathrm{d}S}{\mathrm{d}\xi} = \Lambda \tag{12a}$$

$$\frac{dU}{d\xi} = \Lambda\cos\theta - 1 \tag{12b}$$

$$\frac{\mathrm{d}W}{\mathrm{d}\xi} = \Lambda\sin\theta \tag{12c}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{m}{\Lambda} \tag{13a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}\xi} = \Lambda(P_H\sin\theta - P_V\cos\theta) \tag{13b}$$

$$\frac{\mathrm{d}P_H}{\mathrm{d}\xi} = \Lambda U[K_1 + K_2(U^2 + W^2)] \tag{14a}$$

$$\frac{\mathrm{d}P_V}{\mathrm{d}\xi} = \Lambda W[K_1 + K_2(U^2 + W^2)] \tag{14b}$$

in which

$$\Lambda = (P_H\cos\theta + P_V\sin\theta + \tau)/\lambda^2 + 1 \tag{15}$$

is derived from Eqs. (10) and (11b).

In terms of nondimensional variables, boundary conditions for pinned–pinned and fixed–fixed beams are given below.

$$\text{pinned–pinned:}\ S(0) = 0, \qquad U(0) = 0, \qquad W(0) = 0$$
$$m(0) = 0 \tag{16a}$$

$$U(1) = 0, \qquad W(1) = 0, \qquad m(1) = 0 \tag{16b}$$

$$\text{fixed–fixed:}\ S(0) = 0, \qquad U(0) = 0, \qquad W(0) = 0$$
$$\theta(0) = 0 \tag{17a}$$

$$U(1) = 0, \qquad W(1) = 0, \qquad \theta(1) = 0 \tag{17b}$$

Note that in each case $S(0) = 0$ implies that the length S along the centroidal axis is measured from the left end of the beam. In addition to boundary conditions, the following conditions are imposed to specify a buckled configuration of the beam.

$$\text{pinned–pinned: } \theta(0) = \theta_0 \tag{18}$$

$$\text{fixed–fixed: } m(0) = m_0 \tag{19}$$

Thus, for a specified nonvanishing value of θ_0 or m_0, we can solve Eqs. (12–19) for $(S, U, W, \theta, P_H, P_V, m)$ and the accompanying nondimensional temperature rise τ. If one is not interested in finding the length of the deformed centroidal axis of the beam, then one can ignore Eq. (12a).

III. Solution of the Problem

A. Linear Problem for Buckled Shapes and Buckling Mode Transitions

The onset of buckling of a uniformly heated beam resting on an elastic foundation is governed by the following linear equation obtained from Eqs. (12–14) by setting $\Lambda = 1$, $\sin\theta \approx \theta \approx W'$, $\cos\theta = 1$, and $P_H = -\tau$, and neglecting nonlinear terms in the unknown functions:

$$W'''' + \tau W'' + K_1 W = 0 \tag{20}$$

Here a prime denotes differentiation with respect to ξ. Equation (20) is the same as that of a beam subjected to an axial force [9,25,26] at the ends. The corresponding boundary conditions are

$$\text{pinned–pinned: } W(\xi) = 0, \qquad W''(\xi) = 0, \quad \text{at } \xi = 0 \text{ and } 1 \tag{21}$$

$$\text{fixed–fixed: } W(\xi) = 0, \qquad W'(\xi) = 0, \quad \text{at } \xi = 0 \text{ and } 1 \tag{22}$$

Assuming that $\tau > 2\sqrt{K_1}$, a general solution of Eq. (20) is [9,25]

$$W = C_1 \sin\beta\xi + C_2 \cos\beta\xi + \bar{C}_1 \sin\bar{\beta}\xi + \bar{C}_2 \cos\bar{\beta}\xi \tag{23}$$

in which

$$\beta = \left[\tau/2 + \sqrt{(\tau/2)^2 - K_1}\right]^{1/2}, \qquad \bar{\beta} = \left[\tau/2 - \sqrt{(\tau/2)^2 - K_1}\right]^{1/2} \tag{24}$$

1. *Pinned–Pinned Beams*

For the pinned–pinned beam, assuming that either $C_1 \neq 0$, or $\bar{C}_1 \neq 0$, and recalling that $\beta^2 - \bar{\beta}^2 = \sqrt{\tau^2 - 4K_1} > 0$, Eq. (23) satisfies boundary conditions (21) provided that

$$\sin\beta \sin\bar{\beta} = 0 \tag{25}$$

whose solutions are

$$\text{either } \beta = m\pi, \quad \text{or} \quad \bar{\beta} = n\pi \qquad (m, n = 1, 2, 3, \ldots) \tag{26}$$

Temperatures for different buckling modes, subsequently also called critical temperatures, are given by

$$\tau_m = \frac{K_1}{m^2\pi^2} + m^2\pi^2, \quad \text{or} \quad \tau_n = \frac{K_1}{n^2\pi^2} + n^2\pi^2 \tag{27}$$

The condition for the buckling mode transition is

$$\tau_m = \tau_n \qquad (m \neq n) \tag{28}$$

which gives following values of the elastic foundation parameter

$$(K_1)_{mn} = m^2 n^2 \pi^4 \qquad (m \neq n) \tag{29}$$

and corresponding values of the load parameter are

$$\tau_{mn} = (m^2 + n^2)\pi^2 \qquad (m \neq n) \tag{30}$$

For $m = 1, 2, \ldots, 6$, Fig. 4 shows τ_m/π^2 versus K_1/π^4; each curve is a straight line whose intercept with the vertical axis gives the corresponding temperature at the onset of buckling for $K_1 = 0$. Thus the effect of the linear elastic foundation is to increase the temperature rise required for the onset of buckling. For a fixed buckling mode (i.e., a fixed value of m) the critical temperature depends linearly upon the elastic foundation parameter K_1. The point of intersection, A_{mn}, of two of these curves gives the value of K_1 at which the buckling mode could transition from one to the other. Thus for the linear stiffness parameter and the critical temperature corresponding to the point $[(K_1)_{mn}, \tau_{mn}] = \{m^2(m+1)\pi^4, [m^2 + (m+1)^2]\pi^2\}$, there can be a transition in the buckling mode. Values of the elastic foundation parameter corresponding to the first three transitions in the buckling modes are $4\pi^4$, $64\pi^4$, and $144\pi^4$, which are the same as those given in Wu and Zhong [25], and Hetenyi [27]. When the temperature of the bar is uniformly raised for a fixed value of K_1, the deformed shapes will follow the path indicated by the solid curve in Fig. 4 because it requires a temperature lower than that needed to stay on the original dotted straight line.

If we assume that $C_1 \neq 0$ and $\bar{C}_1 = 0$, then we obtain the following buckled mode shapes:

$$W = C_1 \sin m\pi\xi \qquad (m = 1, 2, 3, \ldots) \tag{31}$$

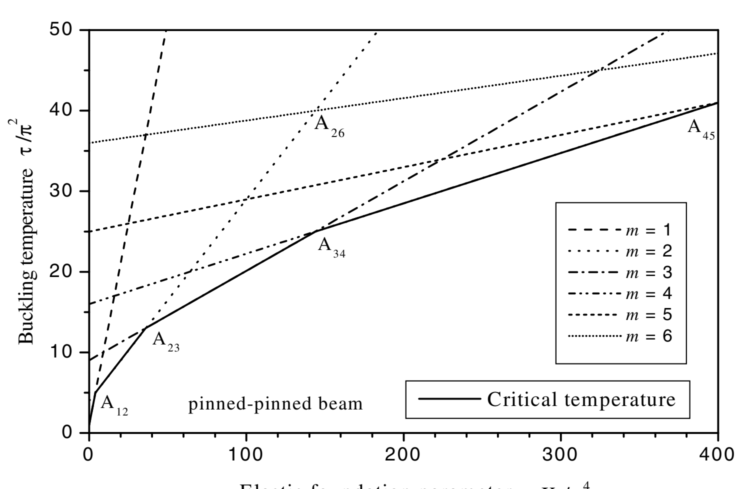


Fig. 4 Dependence of the temperature rise at the onset of buckling upon the foundation stiffness parameter.

The constant C_1 is independent of K_1 implying that buckled mode shapes of a pinned–pinned beam do not depend on the elastic foundation parameter K_1. From Fig. 4, we conclude that the critical buckling mode changes from symmetric (i.e., $m = 1, 3, \ldots$) to antisymmetric (i.e., $m = 2, 4, \ldots$) and vice versa in passing through the points $A_{k(k+1)}$ $(k = 1, 2, 3 \ldots)$ and $A_{2k(2k+1)}$ $(k = 1, 2, 3 \ldots)$, respectively. The order of the critical buckling mode increases with an increase in the value of K_1.

2. *Fixed–Fixed Beams*

For a fixed–fixed beam, we require that the general solution (23) satisfy boundary conditions (22). Recalling that $\tau > 2\sqrt{K_1}$, we arrive at the characteristic equation

$$\tau(1 - \cos\beta\cos\bar{\beta}) = 2\sqrt{K_1}\sin\beta\sin\bar{\beta} \tag{32}$$

which cannot be solved analytically for the load parameter τ as a function of the stiffness parameter K_1 because β and $\bar{\beta}$ depend upon τ and K_1. We thus employ the Newton iteration method to find its roots. Figure 5 depicts the plot of the critical temperature as a function of the linear stiffness parameter of the foundation. Points A_{m1} $(m = 0, 1, 2, 3)$ with coordinates

$$(K_1)_m = m^2(m+2)^2\pi^4, \qquad \tau_m = [m^2 + (m+2)^2]\pi^2 \tag{33}$$

correspond to the buckling mode transitions. As for a pinned–pinned beam, for a given value of the linear stiffness of the foundation, they give the lowest temperature at which buckling will ensue. Thus the three lowest values of K_1 at which buckling mode transitions occur are $9\pi^4$, $64\pi^4$, and $225\pi^4$. The critical temperature versus K_1 curve and the two lowest values of K_1 corresponding to the buckling mode transition coincide with those given in Wu and Zhong [25] for an axially compressed beam. Substitution from Eq. (33) into Eq. (32) gives

$$\beta_m = (m+2)\pi, \qquad \bar{\beta}_m = m\pi \tag{34}$$

For different values of the linear elastic foundation stiffness parameter K_1 we have plotted in Fig. 6 the deformed shape of the beam at the initiation of buckling in one of the first three buckling modes. As also pointed out by Wu and Zhong [25], buckling modes of a fixed–fixed beam depend upon the parameter K_1, and with an increase in the value of K_1, the buckling mode changes from symmetric to antisymmetric, back to symmetric, and so on. However, buckling modes of the pinned–pinned beam do not depend on the value of the linear elastic foundation stiffness parameter K_1.

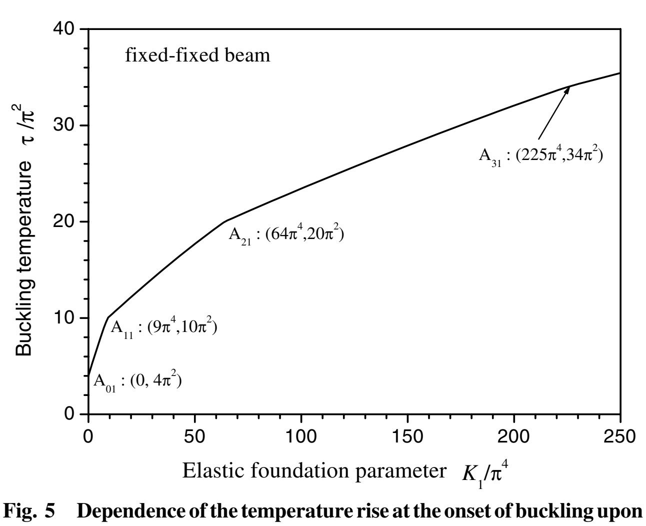
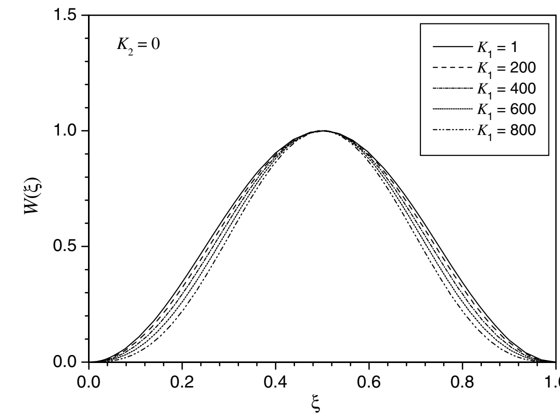
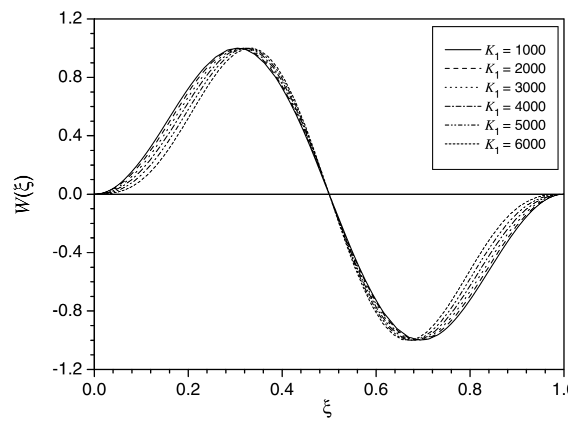


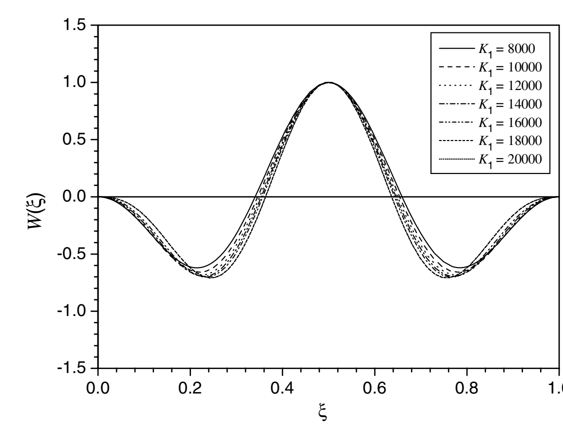
Fig. 5 Dependence of the temperature rise at the onset of buckling upon the linear foundation stiffness parameter.



a) K_1 = 1, 200, 400, 600, 800



b) K_1 = 1000, 2000, …, 6000



c) K_1 = 8000, 10000, 12000, …, 20000

Fig. 6 For specified values of K_1 ($K_2 = 0$), deformed shapes of a fixed–fixed beam at the onset of buckling.

B. Nonlinear Problem for Postbuckled Shapes

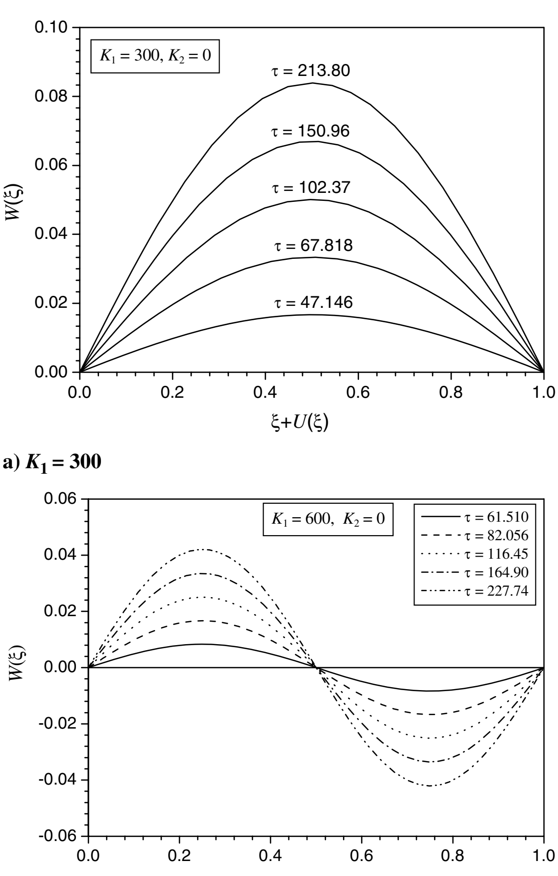
It is difficult to solve analytically the nonlinear coupled boundary-value problem defined by Eqs. (12–19), therefore we find its approximate solution numerically by the shooting method that replaces the two-point boundary-value problem by a sequence of initial-value problems. That is, values of unknown functions at the initial point and unknown parameters are estimated to start computations [26], and these estimates are modified until specified boundary conditions at the terminal point are satisfied; the transformation of the boundary-value problem into an initial-value problem for use in the shooting method is described in the Appendix.

For a pinned–pinned beam, θ_0 is specified, and for a fixed–fixed beam m_0 is specified. Then $P_H(0)$, $P_V(0)$, and τ are varied until Eq. (16b) or Eq. (17b) is satisfied. The Runge–Kutta method is used to integrate Eqs. (12–19) and the Newton–Raphson iteration method is used to compute the varying parameters. By gradually increasing the parameter θ_0 (or m_0), finding the corresponding equilibrium configuration, and using the continuation method, we determine postbuckled configurations of the beam as a continuous function of the temperature rise τ. By assigning the parameter θ_0 (or m_0) a very small value in the computation, we can obtain the critical buckling load and the corresponding mode shape. In computing numerical results presented below we have set $\lambda = 100$.

Figure 7 exhibits the postbuckled configurations obtained by starting from the first (symmetric) and the second (antisymmetric) buckled modes of the pinned–pinned beam for different values of the nondimensional temperature rise τ. For each case the deflection of a point continues to increase monotonically with an increase in the temperature. For a fixed–fixed beam, Fig. 8 depicts the first three postbuckled configurations computed by starting from the buckling modes of Fig. 6. For a prescribed value of K_1 between $4\pi^4$ and $9\pi^4$ that correspond to transition in the buckling modes, equilibrium paths for the two beams in terms of plots of $f = W(0.5)$ versus τ are illustrated in Fig. 9. They are qualitatively similar in the sense that for a fixed value of f, an increase in K_1 requires a higher value of the temperature rise.

To explore the influence of the nonlinear foundation stiffness parameter K_2 on the postbuckling response, we set $K_1 = 200$. Figure 10 shows the variation of the central deflection parameter $f = W(0.5)$ with the temperature rise τ for five values of the stiffness parameter K_2. It is clear from these plots that effects of the nonlinear foundation stiffness parameter are noticeable only for large postbuckling deformations of the beam.

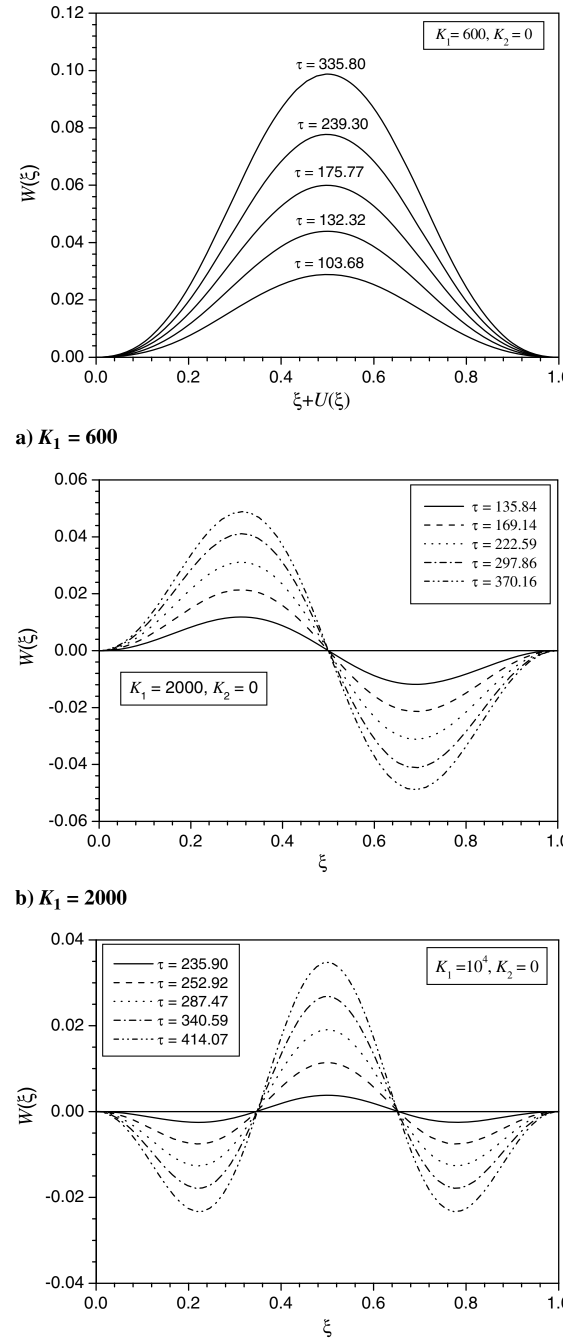
For the fixed–fixed beam and for different values of the stiffness parameter K_1, the variation of the bending moment $m(1)$ with the temperature rise is exhibited in Fig. 11. It shows that the end moment $m(1)$ decreases with an increase in K_1. In contrast to the symmetric buckling of beams without an elastic foundation [22], the end transverse force $P_V(1)$ will be produced when the buckled beam is supported on an elastic foundation. The variations of the transverse force $P_V(1)$ with temperature τ for different values of K_1 are presented in Fig. 12 for both pinned–pinned and fixed–fixed beams. For large values of τ (>60 for the pinned–pinned beam, and >125 for the fixed–fixed beam), the end transverse force $P_V(1)$ increases with an increase in the foundation stiffness parameter K_1.



a) $K_1 = 300$

b) $K_1 = 600$

Fig. 7 For specified values of τ and $K_2 = 0$, postbuckled configurations of a pinned–pinned beam for two values of K_1.



a) $K_1 = 600$

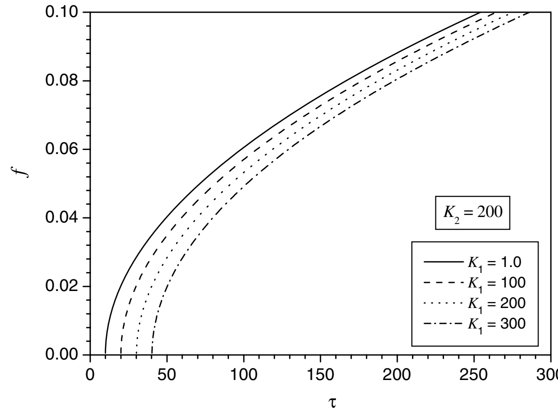
b) $K_1 = 2000$

c) $K_1 = 10^4$

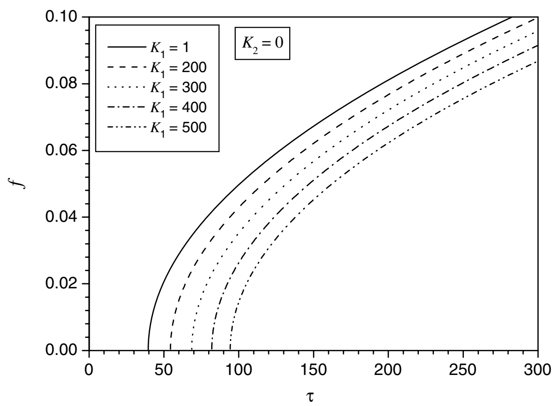
Fig. 8 For specified values of the temperature rise τ, and $K_2 = 0$, postbuckled configurations of a fixed–fixed beam for three values of K_1.

IV. Remarks

A challenging task is to determine experimentally the stiffness of an elastic foundation. One could use the following inverse method to do so. By comparing the deformed shape of the beam for different values of temperature rise with those computed by the present method, one can find the stiffness of the elastic foundation. The challenging task then is to ensure that this value of stiffness will result in the correct temperatures at which buckling mode shapes transition

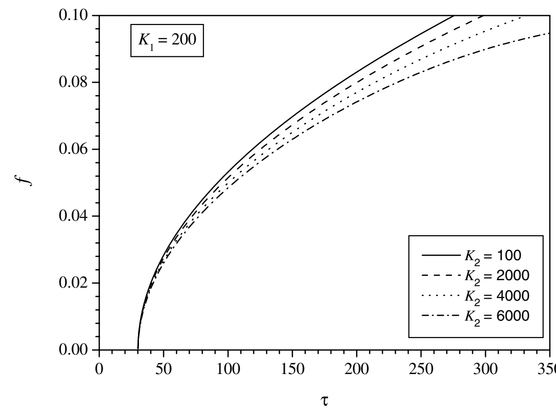


a) Pinned-pinned beam, $K_2 = 200$

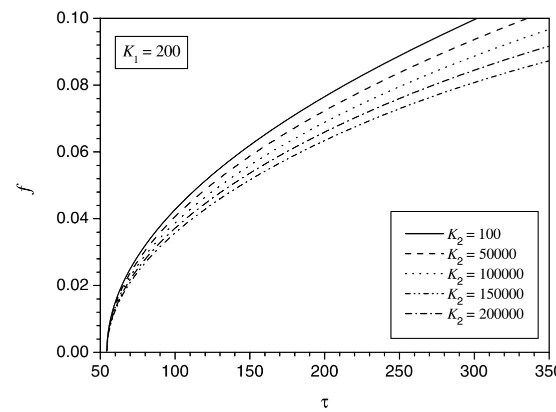


b) Fixed-fixed beam, $K_2 = 0$

Fig. 9 Variation of the dimensionless deflection $f = W(0.5)$ with τ for mode −1 postbuckling deformations of beam.



a) Pinned-pinned beam



b) Fixed-fixed beam

Fig. 10 Dimensionless deflection $f = W(0.5)$ vs τ for beams deformed beyond the critical buckling temperature.

from one to another. Li et al. [28] used a similar procedure to identify the buckled shape of a microresonator.

For the present problem, results computed by taking $\Lambda = 1$ in Eq. (8b) differ from those presented here by less than 0.1% , suggesting thereby that the consideration of the stretching of the midsurface of the beam in Eq. (8b) does not have much influence on the postbuckled shapes computed here. However, for a microelectromechanical beam, Batra et al. [29] used the von Karman approximation to consider stretching of the beam's midsurface and found that it significantly affects the pull-in voltage. Tiersten [30] has recently revisited the Euler–Bernoulli beam theory and delineated various approximations made to simplify the problem. Several investigators [5,9,31] have set $\Lambda = 1$ in Eq. (8b) while studying postbuckling deformations of beams.

V. Conclusions

We have studied the buckling and the postbuckling deformations of uniformly heated pinned–pinned and fixed–fixed Euler–Bernoulli beams supported on nonlinear elastic foundations. The geometric nonlinearity introduced by the curvature of the deformed beam, and the constraint force of the elastic foundation in both longitudinal and transverse directions, are incorporated in the problem formulation. Buckling modes and transitions among them are computed by solving analytically the linear problem. The dependence of the initiation of the first few buckling modes upon the linear foundation stiffness parameter is plotted from which values of the Winkler foundation parameter corresponding to the transition in the buckling mode are determined. The nonlinear boundary-value problem for postbuckling deformations is solved by the shooting method by first transforming it to an initial-value problem. For different values of the elastic foundation parameter, equilibrium paths and configurations derived from the first buckling mode are illustrated. As expected, the nonlinear foundation stiffness parameter does not influence the buckling temperature, and has a small effect on the postbuckling deformations as compared with the effect of the linear foundation stiffness parameter. The problem studied here is directly related to the compressed elastic column. The analysis presented here applies to buckling and postbuckling deformations of a hygroscopic beam supported on an elastic foundation, and exposed to moisture.

Appendix: Shooting Method for the Nonlinear Problem

We briefly describe the procedure used to numerically solve the nonlinear two-point boundary-value problem defined by Eqs. (12–19). We write it as

$$\frac{d\boldsymbol{Y}}{d\xi} = \boldsymbol{H}(\xi, \boldsymbol{Y}), \qquad (0 < \xi < 1) \tag{A1a}$$

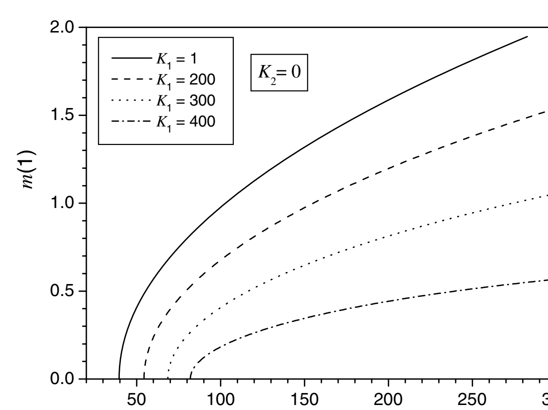
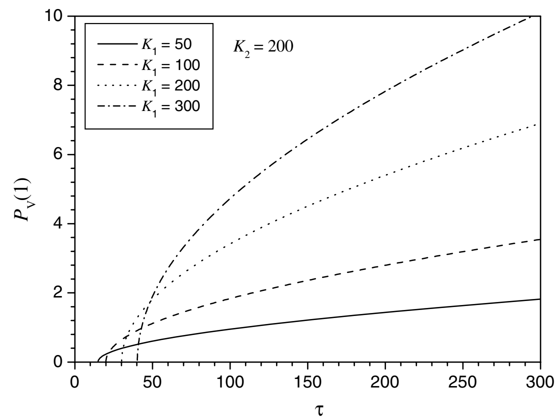
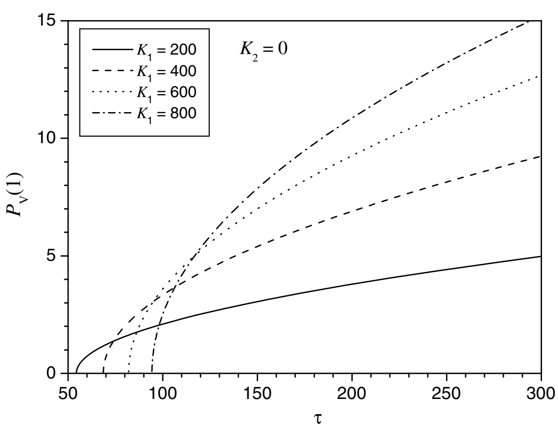


Fig. 11 Dependence of the bending moment $m(1)$ upon τ for beams deformed beyond the critical buckling temperature.



a) Pinned-pinned beam



b) Fixed-fixed beam

Fig. 12 Dependence of the $P_V(1)$ upon the nondimensional temperature rise τ for beams deformed beyond the critical buckling mode.

$$\boldsymbol{B}_0\boldsymbol{Y}(0) = \boldsymbol{b}_0 \tag{A1b}$$

$$\boldsymbol{B}_1\boldsymbol{Y}(1) = \boldsymbol{b}_1 \tag{A1c}$$

in which

$$\boldsymbol{Y} = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}^T = \{S, U, W, \theta, m, P_H, P_V, \tau\}^T$$

$$\boldsymbol{H} = \{\Lambda, \Lambda\cos y_4 - 1, \Lambda\sin y_4, -y_7/\Lambda, F_1, F_2, F_3, 0\}^T$$

$$F_1 = \Lambda(y_5\sin y_4 - y_6\cos y_4), \qquad F_2 = \Lambda y_2\Big[K_1 + K_2\Big(y_2^2 + y_3^2\Big)\Big]$$

$$F_3 = \Lambda y_3\Big[K_1 + K_2\Big(y_2^2 + y_3^2\Big)\Big]$$

$$\Lambda = (y_5\cos y_4 + y_6\sin y_4 + y_8)/\lambda^2 + 1, \qquad \boldsymbol{b}_1 = \{0, 0, 0\}^T$$

$$\boldsymbol{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b}_0 = \{0, 0, 0, \theta_0, 0\}^T$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for pinned-pinned beam}$$

$$\boldsymbol{b}_0 = \{0, 0, 0, 0, m_0\}^T$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{for fixed-fixed beam}$$

We consider the following initial-value problem corresponding to the boundary-value problem (A1):

$$\frac{\mathrm{d}\boldsymbol{Z}}{\mathrm{d}\xi} = \boldsymbol{H}(\xi, \boldsymbol{Z}), \qquad (\xi > 0) \tag{A2a}$$

$$\boldsymbol{Z}(0) = \boldsymbol{I}(\boldsymbol{D}) \tag{A2b}$$

in which $\boldsymbol{Z} = \{z_1, z_2, \ldots, z_8\}^T$, and $\boldsymbol{D} = \{d_1, d_2, d_3\}^T$ is an unknown vector related to initial values of functions $z_6(\xi)$, $z_7(\xi)$, and $z_8(\xi)$; $\boldsymbol{I}(\boldsymbol{D}) = \{0, 0, 0, \theta_0, m_0, d_1, d_2, d_3\}^T$ with $m_0 = 0$ for a pinned–pinned beam, and $\theta_0 = 0$ for a fixed–fixed beam. A solution of the initial-value problem (A2) may be symbolically written as

$$\boldsymbol{Z}(\xi; \theta_0, m_0, \boldsymbol{D}) = \boldsymbol{I}(\boldsymbol{D}) + \int_0^{\xi} \boldsymbol{H}(\eta, \boldsymbol{Z})\mathrm{d}\eta \tag{A3}$$

For a given nonzero value of θ_0, or m_0 (one of them must be zero), we find that solution of (A3) which satisfies boundary condition (A1c), that is,

$$\boldsymbol{B}_1\boldsymbol{Z}(1; \theta_0, m_0, \boldsymbol{D}) = \boldsymbol{b}_1 \tag{A4}$$

Obviously, if $\boldsymbol{D} = \boldsymbol{D}^*$ is a root of Eq. (A4), then a solution of the boundary—value problem (A1) is

$$\boldsymbol{Y}(\xi) = \boldsymbol{Z}(\xi; \theta_0, m_0, \boldsymbol{D}^*) \tag{A5}$$

We employ the Runge–Kutta method to integrate the system (A2) of ordinary differential equations, and the Newton–Raphson iteration method to search for a root $\boldsymbol{D}^*$ of Eq. (A5) to find a numerical solution of the boundary-value problem (A1).

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B. Balachandran
Associate Editor