Material tailoring for reducing stress concentration factor at a circular hole in a functionally graded material (FGM) panel

G.J. Nie*,b,a, Z. Zhonga, R.C. Batrab

*a School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China
b Department of Biomedical Engineering and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

ABSTRACT

By assuming that Young’s modulus and Poisson’s ratio of a linearly elastic and isotropic material vary along the radial direction in a panel with a circular hole and deformed by a far field uniaxial tensile traction, we first analytically find the stress concentration factor, K, at the hole. The problem is solved by superposing solutions of two problems – one of uniform biaxial tension and the other of pure shear. The solutions of the first and the second problem are, respectively, in terms of hypergeometric functions and Frobenius series. Subsequently, we analytically study the material tailoring problem for uniform biaxial tension, and give explicit variation of Young’s modulus to achieve a prespecified K. For the panel loaded by a far field uniaxial tensile traction, we show that the K can be reduced by a factor of about 8 by appropriately grading Young’s modulus and Poisson’s ratio in the radial direction. By plotting K versus the two inhomogeneity parameters, we solve the material tailoring problem for a panel loaded with a far field uniaxial traction. The analytical results should serve as benchmarks for verifying the accuracy of approximate/numerical solutions for an inhomogeneous panel.

1. Introduction

Even though the mechanical behavior of an inhomogeneous material has been studied since 1950’s, there has been tremendous activity in this field during the last three decades [1–6]. A heterogeneous material with continuous spatial variation of material parameters is often called a functionally graded material (FGM). With the availability of 3-D printing for manufacturing materials with complex microstructures, it is now feasible to fabricate structures to have the optimum stress and strain distributions for enhancing their mechanical properties under prescribed loads [7,8]. One such problem is controlling the stress concentration factor, K, around a circular hole in a panel.

It is well known that K at a circular hole in an infinite panel composed of a homogeneous, isotropic and linearly elastic material deformed in uniaxial tension equals 3 [9]. For orthotropic materials Lekhnitskii et al. [10] have deduced K for an infinite plate containing a circular hole and deformed by remote uniaxial tensile tractions. Based on Lekhnitskii’s solution of the plane elastostatics problem using complex variables, Britt [11], Tenchev et al. [12] and Xu et al. [13,14], respectively, found K for anisotropic rectangular panels with centrally located circular and elliptical cutouts, laminated composites with circular holes, and composite laminates with either an elliptical hole or multiple holes.

Authors of Refs. [15–18] employed the finite element method (FEM) to evaluate stresses in composite laminates with circular holes. Kubair and Bhanu-Chandar [19] and Enab [20] using the FEM found that K is significantly influenced by the spatial variation of the material inhomogeneity. One needs a very fine mesh near the hole and conduct convergence studies to deduce reasonably accurate values of K that can be an arduous task.

Huang and Haftka [21], and Cho and Rowlands [22,23] optimized fiber orientations near a hole to minimize K and increase the load-carrying capacity of composite laminates. Lopes et al. [24] and Gomes et al. [25] found fiber orientation angles and their volume fractions either to minimize the peak stress around cutouts or to maximize the buckling and the first-ply failure load of composite panels. They pointed out that the optimum variable-stiffness designs with a central hole can have nearly the same initial buckling loads as panels with the same volume fractions of fibers but no hole.

By dividing an inhomogeneous material panel into a series of piecewise homogeneous layers and using the method of complex variables, Yang et al. [26,27], Yang and Gao [28] and Kushwaha and Saini [29] have shown that when Young’s modulus decreases with the distance from the hole boundary, K > 3. Mohammadi et al. [30] analytically found K around a circular hole in an infinite FGM plate subjected to uniform biaxial tension and pure shear by assuming that both Young’s
modulus and Poisson’s ratio vary exponentially in the radial direction. By assuming that Young’s modulus has a power law variation in the radial direction and Poisson’s ratio is a constant, Sburlati [31] studied the elastic response of an FGM annular ring inserted in a hole of a homogeneous plate. Kubair [32] used the method of separation of variables to find closed-form expressions for stresses and displacements in FGM plates with and without holes under anti-plane shear loading and used a non-traditional definition of $K$.

We note that there are a limited number of analytical studies on the stress concentration around a circular hole in isotropic FGM panels. Furthermore, there are no results on material tailoring for reducing stress concentration around a circular hole in isotropic FGM panels. We analytically find $K$ at a circular hole in an isotropic FGM panel under a far field uniaxial tensile traction, (ii) analyze the material tailoring problem for uniform biaxial tension loading, and (iii) investigate the effect of material inhomogeneity parameters on $K$. For far field uniform tensile loading, we provide a response function to estimate inhomogeneity parameters for a desired value of $K$.

The rest of the paper is organized as follows. Sections 2 and 3, respectively, give the formulation and the solution of the direct problem in which we analyze deformations of the panel under prescribed far field surface tractions. Section 3 is divided into three subsections that provide details of deformations under uniform biaxial tension, pure shear and uniaxial tension, respectively. In Section 4 we analytically solve the material tailoring problem for uniform biaxial tension loading. Section 5 provides numerical results that establish the accuracy and the convergence of the series solution for the pure shear loading, and delineate effects of the variation of the material properties on $K$ and stress distributions. Conclusions of the work are summarized in Section 6.

2. Formulation of the direct problem

We consider an isotropic and linearly elastic FGM panel with a circular hole of radius $a$ subjected to a far-field uniaxial traction $\sigma_r$, as shown in Fig. 1(a). We analyze how the radial variation in Young’s modulus and Poisson’s ratio affects the stress concentration at the hole periphery for plane stress deformations of the panel. We solve the problem by superposing solutions of two problems – biaxial tension and pure shear, as shown in Fig. 1(b) and (c).

We use cylindrical coordinates $(r, \theta)$ with origin at the hole center to describe the panel deformations. In the absence of body forces, equilibrium equations are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (r \sigma_\theta) + \frac{\partial \sigma_\theta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (r \sigma_r) = 0,$$

$$\frac{\partial \sigma_\theta}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (r \sigma_\theta) + \frac{2}{r} \sigma_r = 0,$$  

(1a,b)

where $\sigma_r$, $\sigma_\theta$ and $\sigma_\phi$ are stress components. Hooke’s law relating stresses to infinitesimal strains, $\varepsilon_r$, $\varepsilon_\theta$, $\varepsilon_\phi$, is

$$\varepsilon_r = \frac{1}{E(r)} \left[ \sigma_r - \nu(r) \sigma_\theta \right], \quad \varepsilon_\theta = \frac{1}{E(r)} \left[ \sigma_\theta - \nu(r) \sigma_r \right], \quad \varepsilon_\phi = \frac{2(1 + \nu(r))}{E(r)} \sigma_\phi.$$  

(2a-c)

We assume that Young’s modulus, $E(r)$, and Poisson’s ratio, $\nu(r)$, are given by either

(i) general power-law variations

$$E(r) = E_0 \left[ 1 + \beta_1 \left( \frac{r}{a} \right)^n \right], \quad \nu(r) = \nu_0 \left[ 1 + \beta_2 \left( \frac{r}{a} \right)^n \right], \quad \text{with } n < 0$$  

(3a)

or

(ii) exponential and power-law variations

$$E(r) = E_0 \exp \left[ \gamma_1 \left( \frac{r}{a} \right)^{-1} \right], \quad \nu(r) = \nu_0 \left[ 1 + \gamma_2 \left( \frac{r}{a} \right)^{-1} \right],$$  

(3b)

where $E_0 = \lim_{r \to a} E(r)$ and $\nu_0 = \lim_{r \to a} \nu(r)$, $\beta_1$, $\beta_2$ and $\gamma_1$, $\gamma_2$ ($-1 < \beta_1, \beta_2, \gamma_1, \gamma_2 < 1$) are real numbers, and $n$ is a positive integer which helps find an analytical solution of the problem. For a homogeneous material, $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$. The variations of $E$ with $r/a$ for $n = -3, -5$ and $\beta_1 = 0.9, -0.9$ depicted in Fig. 2(a) reveal that $E(r)/E_0 \to 1$ as $r/a \to 5$. Similarly, the variation of $E$ with $r/a$ for $\gamma_1 = 0.9, -0.9$ depicted in Fig. 2(b) implies that $E(r)/E_0 \to 1$ as $r/a \to 50$.

The far-field boundary conditions for the biaxial tension and the pure shear problems are:

$$\lim_{r \to \infty} \sigma_r(r) = \frac{\sigma_0}{2},$$  

(4a)

$$\lim_{r \to \infty} \sigma_r(r, \theta) = \frac{\sigma_0}{2} \cos 2\theta, \quad \lim_{r \to \infty} \sigma_\theta(r, \theta) = -\frac{\sigma_0}{2} \sin 2\theta$$  

(4b)

and boundary conditions at the hole periphery are

$$\sigma_r(a, \theta) = 0, \quad \sigma_\theta(a, \theta) = 0.$$  

(4c)

When solving the problem for stresses, we employ the following compatibility equation:

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ r \frac{\partial \varepsilon_\theta}{\partial r} \right] + \frac{1}{r^2} \frac{\partial^2 \varepsilon_\varphi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \varepsilon_\theta}{\partial \varphi^2} = \frac{1}{r} \frac{\partial \varepsilon_r}{\partial r} + \frac{1}{r} \frac{\partial \varepsilon_\theta}{\partial \theta} + \frac{1}{r^2} \frac{\partial \varepsilon_\phi}{\partial \varphi}.$$  

(5)

3. Solution of the direct problem

We analytically solve the problem for the uniform biaxial tension in subsection 3.1, and use the Frobenius series to analyze the problem for pure shear loading in subsection 3.2. By superposing solutions of these two problems, we obtain the solution for the uniaxial tension problem in subsection 3.3.

3.1. Uniform biaxial tension

We note that the problem geometry, the material properties and the
far field loading are axisymmetric. Accordingly, we assume panel’s deformations to be axisymmetric for which equilibrium Eq. (1) reduces to

$$\sigma_{\theta\theta} = \sigma_r + \frac{d\sigma_r}{dr}$$

(6)

Thus knowing $\sigma_r$, one can find $\sigma_{\theta\theta}$ from Eq. (6).

We introduce a non-dimensional parameter, $\rho = \frac{r}{a}$, and substitute for strains from Eq. (2) into the compatibility condition (5) to get

$$d^2\sigma_r + \left( \frac{3}{\rho} - \frac{E'}{E} \right) \frac{d\sigma_r}{dr} + \frac{vE'}{\rho E} \frac{dE}{d\rho} - \frac{E' - \nu'}{\rho} \sigma_r = 0$$

(7)

where $\left( \frac{\gamma}{\rho} \right)' = \frac{d\gamma}{d\rho}(\cdot)$.

### 3.1.2. Exponential and power-law variations of $E(r)$ and $v(r)$ given by Eq. (3b)

Substituting for $E(r)$ and $v(r)$ from Eq. (3b) into Eq. (7) and solving the resulting equation, we get

$$\sigma_r = \frac{C_1}{\rho^2} + C_2, \quad \text{for } \beta_1 = 0, \ \beta_2 = 0$$

(8a)

$$\sigma_r = \frac{C_1}{\rho^2} jF_i(t_1, t_2; t_3; -\beta_1 \rho^2) + C_2 jF_i(t_4, t_5; t_6; -\beta_2 \rho^2), \quad \text{for } n \leq -3$$

(8b)

$$\sigma_r = C_1 \left( \frac{\rho}{\beta_1} \right)^n jF_i(t_1, t_2; t_3; -\beta_1) + C_2 \left( \frac{\rho}{\beta_1} \right)^n jF_i(t_4, t_5; t_6; -\beta_2),$$

for $n = -1$.

$$\sigma_r = C_1 \left( \frac{\rho}{\beta_1} \right)^n jF_i(t_1, t_2; t_3; -\beta_1) + C_2 \left( \frac{\rho}{\beta_1} \right)^n jF_i(t_4, t_5; t_6; -\beta_2),$$

for $n = -2$.

Here $C_1$ and $C_2$ are constants to be determined from boundary conditions (4a) and (4c). $jF_i(a; b; c; z) = \sum_{n=0}^{\infty} (a)_n (b)_n (b)_n z^n / n!$ is a hypergeometric function, and

$$t_1 = H_2 - H_1, \ t_2 = H_2 + H_3, \ t_3 = 1 - \frac{2}{n}, \ t_4 = H_1 - H_3, \ t_5 = H_1 + H_3, \ t_6 = 1 + \frac{2}{n}, \ t_7 = -\frac{3}{2} H_4, \ t_8 = \frac{1}{2} H_4, \ t_9 = -\frac{3}{2} H_4, \ t_{10} = \frac{1}{2} H_4$$

$$t_{11} = -1 - H_2, \ t_{12} = -H_2, \ t_{13} = 1 - 2 H_2, \ t_{14} = -1 + H_2, \ t_{15} = 1 + 2 H_2, \ t_{16} = \frac{1}{n + \frac{1}{2}} + \frac{1}{2 n},$$

$$H_1 = -\frac{1}{4}, \ H_2 = -\frac{1}{2}, \ H_3 = 1 - \frac{1}{2}, \ H_4 = -\frac{1}{2}, \ H_5 = 1 + \frac{1}{2}, \ H_6 = \frac{1}{2}$$

(9)

For $n_1 = 0$, the solution can be expressed in terms of Bessel functions $J_n(z)$ and $Y_n(z)$, respectively, of the first and the second kind, and the modified Bessel functions $I_n(z)$ and $K_n(z)$.

We note that the solution (8a) for the homogeneous material agrees with that given in Timoshenko’s book [9]. Constants $C_i$, $i = 1, 2, \ldots$ appearing in different equations do not, in general, have the same values.

### 3.2. Pure shear deformations

We introduce an Airy stress function, $\varphi(r, \theta)$, and note that

$$\sigma_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2}, \ \sigma_{\theta\theta} = \frac{\partial^2 \varphi}{\partial \rho^2}, \ \sigma_{\rho\theta} = -\frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial \varphi}{\partial \rho} \right)$$

(12)

identically satisfy equilibrium Eq. (1). In view of far-field condition (4b), we assume that

$$\varphi(r, \theta) \sim \varphi_0(r) \cos \theta$$

(13)

From Eqs. (2), (5), (12) and (13), we find that the compatibility condition reduces to

$$d^2 \varphi_1 \frac{d^2 \varphi_1}{d\rho^2} + z_1(\rho) d^2 \varphi_1 \frac{d^2 \varphi_1}{d\rho^2} + z_2(\rho) d^2 \varphi_1 \frac{d^2 \varphi_1}{d\rho^2} + z_3(\rho) \frac{d \varphi_2}{d\rho} + z_4(\rho) \frac{d \varphi_2}{d\rho} = 0$$

(14)

where

$$z_0(\rho) = -\frac{4 v E' \rho^2 + 8 v (E')^2}{\rho^2 E^2} + \frac{8 v E' \rho^2}{\rho^2 E^2} + \frac{12 v E' \rho^2}{\rho^2 E^2} + \frac{4 v E'}{\rho^2},$$

$$z_1(\rho) = \frac{v E'}{\rho^2} \left( 2 \frac{v (E')^2}{\rho^2 E^2} + \frac{8 v E' \rho^2}{\rho^2 E^2} + \frac{9 v E'}{\rho^2} \right),$$

$$z_2(\rho) = \frac{-E^*}{E} + \frac{2 (E')^2}{E^2} + \frac{8 v E' \rho^2}{\rho^2 E^2} - \frac{9 v}{\rho^2} z_3(\rho) \rho,$$

$$z_3(\rho) = \frac{2}{\rho} \frac{2 E^*}{E}$$

(15)

We solve the 4th order ordinary differential Eq. (14) with variable coefficients by the method of Frobenius series.
3.2.1. General power-law variations of $E(r)$ and $v(r)$ given by Eq. (3a)

We assume that

$$\varphi_2(r) = \rho^r \sum_{k=0}^{M} b_k(s)(\rho^n)^k, \quad \text{with } b_0(s) = 1$$  \hspace{1cm} (16)

where $b_k, k = 0, 1, 2, \ldots, M$, are given by the recurrence formulae, the variable $s$ is determined, and $M$ equals the total number of terms in the series.

Substituting from Eqs. (3a) and (16) into Eq. (14) and equating to zero coefficients of different powers of $\rho$, we get recurrence formulae ((17a) and (17b)) for $b_k$ and the indicial Eq. (17c) for $s$:

$$b_1(s) = \frac{B_1(s)b_0(s)}{(n + s - 4)(n + s - 2)(n + s)}$$  \hspace{1cm} (17a)

$$b_{k+2}(s) = \frac{B_2(s)b_0(s) + B_1(s)b_k(s)}{(nk + 2n + s - 4)(nk + 2n + s - 2)(nk + 2n + s)}$$  \hspace{1cm} (17b),\quad k = 0, 1, 2, \ldots, M - 2

$$s(s - 4)(s - 2) = 0$$  \hspace{1cm} (17c)

The lengthy expressions for $B_k(s)$ ($i = 1, 2, 3$) are listed in Appendix A.

Roots, $s = 4, 2, 0, -2$, of the indicial Eq. (17c) are independent of the inhomogeneity parameters $\beta_0, \beta_1, \beta_2$ and $n$. According to the Frobenius method [33], solutions of the stress function for different values of $n$ are

$$\varphi_2(r) = C_0 \rho^0 \sum_{k=0}^{M} b_k(s)(\rho^n)^k + C_1 \rho^1 \sum_{k=0}^{M} b_k(s)(\rho^n)^k$$

$$+ C_2 \rho^2 \frac{d^2b_0(s)}{ds^2} + 2\ln\rho \frac{db_0(s)}{ds} + (\ln\rho)^2b_0(s)$$  \hspace{1cm} (18a)

$$< 0 \text{ and } n \neq -1, n \neq -2, n \neq -4$$

$$\varphi_2(r) = C_3 \rho^3 \sum_{k=0}^{M} \frac{d^3b_k(s)}{ds^3} + 2\ln\rho \frac{db_k(s)}{ds} + (\ln\rho)^2b_k(s)$$  \hspace{1cm} (18b)

$$< 0 \text{ and } n = -1, n = -2$$

$$\varphi_2(r) = C_3 \rho^3 \sum_{k=0}^{M} \frac{d^3b_k(s)}{ds^3} + 2\ln\rho \frac{db_k(s)}{ds} + (\ln\rho)^2b_k(s)$$  \hspace{1cm} (18c)

where $b_k(s) = (s - 2)^k b_k(s)$ and $C_0, c_1 = 1, 2, 3, 4$, are constants to be determined from boundary conditions (4b) and (4c).

3.2.2. Exponential and power-law variations of $E(r)$ and $v(r)$ given by Eq. (3b)

For $E(r)$ and $v(r)$ given by Eq. (3b) we also use the method of Frobenius series to solve Eq. (14). Letting $n = -1$ in Eq. (16), we have

$$\varphi_2(r) = \rho^r \sum_{k=0}^{M} b_k(s)(\rho^{-1})^k, \quad \text{with } b_0(s) = 1$$  \hspace{1cm} (19)

Substituting from Eqs. (3b) and (19) into Eq. (14), and equating to zero coefficients of different powers of $\rho$, we get following recurrence formulae ((20a), (20b), (20c)) for $b_k$ and the indicial equation (20d) for $s$:

$$b_1(s) = \frac{D_1(s)b_0(s)}{(s - 5)(s - 3)(s - 1)}$$  \hspace{1cm} (20a)

$$b_1(s) = \frac{D_1(s)b_0(s) + D_2(s)b_1(s)}{s(s - 6)(s - 4)}$$  \hspace{1cm} (20b)

$$b_{k+1}(s) = \frac{D_1(s)b_0(s) + D_2(s)b_{k+1}(s) + D_3(s)b_{k+2}(s)}{(k + 1 - s)(k + 3 - s)(k + 5 - s)(k + 7 - s)}, \quad k = 0, 1, 2, \ldots, M - 3$$  \hspace{1cm} (20c)

$$s(s - 4)(s - 2)(s + 2) = 0$$  \hspace{1cm} (20d)

The lengthy expressions for $D_k(s)$ ($i = 1, 2, 3$) are listed in Appendix B.

The indicial Eq. (20d) is the same as the indicial Eq. (17c) and is independent of the inhomogeneity parameters $\gamma_0, \gamma_2$. Thus

$$\varphi_2(r) = C_1 \rho^1 \sum_{k=0}^{M} \left[ \frac{d^2 b_k(s)}{ds^2} + 2\ln\rho \frac{db_k(s)}{ds} + (\ln\rho)^2b_k(s) \right]$$

$$+ C_1 \rho^1 \sum_{k=0}^{M} \left[ \frac{d^2 b_k(s)}{ds^2} + \ln\rho \frac{db_k(s)}{ds} + (\ln\rho)^2b_k(s) \right]$$  \hspace{1cm} (21)

where $b_k(s) = (s - 2)^2 b_k(s)$ and $C_1, i = 1, 2, 3, 4$, are constants to be determined from boundary conditions (4b) and (4c).

From the expression (Eq. 18) or (Eq. 21) of the stress function $\varphi_2(r)$, we find stresses by using Eqs. (12) and (13).

3.3. Stress concentration factor, $K$, for the uniaxial tension problem

We note that for the uniform biaxial tension and the pure shear problems, $\sigma_0$ has the maximum value at $\theta = \frac{\pi}{6}$. Denoting by $K_0, K_2$ and $K_3$, respectively, the stress concentration factor corresponding to the biaxial tension, the pure shear and the uniaxial tension problems (see Fig. 1), we have [34]

$$K_0 = \frac{2\sigma_0(\gamma)_{\gamma=1}}{\sigma_0}, \quad K_2 = \frac{2\sigma_0(\gamma, \theta)_{\gamma=1, \theta=0, \pi/6}}{\sigma_0}, \quad K_3 = \frac{K_0 + K_2}{2}$$  \hspace{1cm} (22a, b, c)

4. Analytical solution of material tailoring for uniform biaxial tension

We can analytically solve the material tailoring problem for uniform biaxial tension loading but not for the simple shear problem since its solution is in terms of Frobenius series. Assume that the stress state is

$$\sigma_{\theta} = \frac{C_0}{2}(1 - \rho^m), \quad \sigma_{\phi} = \frac{C_0}{2}(1 - (m + 1)ho^m) \quad \text{with } m < 0$$  \hspace{1cm} (23)

For $m = -1$, $\sigma_{\theta} = \frac{C_0}{2}$ is a constant in the panel and equals the far field axial stress. Also, the boundary condition at the hole periphery is exactly satisfied. For the stress state shown in Eq. (23), Eq. (22a) gives $K_0 = -m > 0$  \hspace{1cm} (24)

We now find $E(r)$ and $v(r)$ to achieve $K_0 = -m$. Substituting for the stress $\sigma_{\theta}$ from Eq. (23) into Eq. (7), we obtain

$$[\rho^m(m + 1) - 1] \frac{E(r)}{E(r)} - m\rho^m - m\rho^{m-1} = 0$$  \hspace{1cm} (25)

We have one ordinary differential equation for two unknown
functions, $E(\rho)$ and $v(\rho)$. One way to solve the problem is to assume polynomial expressions for $E(\rho)$ and $v(\rho)$, and find unknowns in them by minimizing the value of the square of the left hand side of Eq. (25). Here, we assume that $v(\rho) = v_0 = a$ constant, and solve Eq. (25) for $E(\rho)$. Eq. (25) has the solution

$$E(\rho) = E_0\left(\frac{m + 1}{m}\right)^{\rho m + v_0(1-\rho m) - 1} \right)^{\frac{m+2}{m}}$$

(26)

where $E_0 = E(\rho)|_{\rho=1}$. We note that

$$E_{\text{ex}} = \lim_{\rho \to 0} E(\rho) = E_0\left(\frac{m+1}{m}\right)^{\frac{m+2}{m}}$$

(27)

Thus, for $m = -2$, Eqs. (24) and (26) gives $E(\rho) = E_0$ and $K_0 = 2$ which equals that for a homogeneous material.

For $K_0 = 1, 2, 3$ and 4, the required variations of $E(\rho)/E_0$ taking $v_0 = 0.25$ displayed in Fig. 3, respectively, have $E_{\text{ex}}/E_0 = 3.161, 1.0, 0.54, 0.357$. Thus $\frac{dE(\rho)}{d\rho} |_{\rho=1} > 0$ ($< 0$) for $K_0 < 2$ (> 2).

5. Numerical results

We first establish in subsection 5.1 that the Frobenius series solution converges to an analytical solution of the problem, and then in subsection 5.2 combine it with the analytical solution of the biaxial tension problem to get a solution of the far field simple tensile loading. We use it in subsection 5.2.2 to ascertain the effect of material inhomogeneity parameters on the stress concentration factor, $K$.

5.1. Accuracy and convergence of the solution for the pure shear problem

We first find the number of terms in the Frobenius series solution that give a converged solution for the pure shear problem. We note that when only $v$ varies, i.e., $\beta_0 = 0$ in Eq. (3a) or $\gamma = 0$ in Eq. (3b), the analytical solution (14) for the general power-law variations of $E(r)$ and $v(r)$ given by Eq. (3a) is

$$\varphi_1(\rho) = C_1 \left(\frac{v_0 \beta_0 \rho^n}{n}\right) 2 F_3(d; d; d; \frac{v_0 \beta_0 \rho^n}{n})$$

$$+ C_2 \left(\frac{v_0 \beta_0 \rho^n}{n}\right)^2 2 F_3(d; d; d; \frac{v_0 \beta_0 \rho^n}{n}) + C_3 2 F_3(d; d; d; \frac{v_0 \beta_0 \rho^n}{n})$$

(28a)

The solution for the exponential and the power-law variations of $E(r)$ and $v(r)$ given by Eq. (3b) is

$$\varphi_2(\rho) = C_1 \left(\frac{v_0^{\gamma} \rho^{\gamma}}{\rho}\right)^2 2 F_3(d; d; d; \frac{v_0^{\gamma} \rho^{\gamma}}{\rho}) + C_2 G_1^{(2)}(\frac{-v_0^{\gamma} \rho^{\gamma}}{\rho}) d_1, d_2$$

$$+ C_3 G_2^{(2)}(-\frac{v_0^{\gamma} \rho^{\gamma}}{\rho}) \left| d_1, d_2 \right|$$

$$+ C_4 G_3^{(2)}(-\frac{v_0^{\gamma} \rho^{\gamma}}{\rho}) \left| d_1, d_2 \right|$$

(28b)

where constants $C_i$, $i = 1, 2, 3, 4$, are determined from boundary
conditions (4b) and (4c), \( G_{M_{nm}}^{M_{nm}}(c) \) is the Meijer G function, and 
\[ d_1 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_2 = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
and 
\[ d_1 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_2 = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
\[ d_3 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_4 = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
\[ d_5 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_6 = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
\[ d_7 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_8 = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
\[ d_9 = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_{10} = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]
\[ d_{11} = \left\{ \frac{1}{2} + \frac{3}{2} - N, \frac{1}{2} + \frac{3}{2} + N \right\}, d_{12} = \left\{ 1 + \frac{3}{2}, 1 + \frac{3}{2} + N \right\}, \]

For a homogeneous material, i.e., \( \beta_1 = \beta_2 = 0 \) in Eq. (3a) or \( \gamma_1 = \gamma_2 = 0 \) in Eq. (3b), the solution (14) is 
\[ \varphi_2(\rho) = C_1 \rho^4 + C_2 \rho^2 + C_3 + C_4 \rho^2 \]
(29)

One gets the stress concentration factor, \( K_4 = 4 \), for the pure shear problem.

Values of \( K_4 \) listed in Table 1 for different values of \( M \) suggest that

M = 5 in the series solution gives a reasonably accurate value of \( K_4 \), and the series solution rapidly converges for the considered values of \( n, \beta_1 \), and \( \beta_2 \).

For general power-law variations of \( E(r) \) and \( v(r) \) given by Eq. (3a), values of \( K_4 \) listed in Table 2 for different values of \( \beta_1 \) computed using \( M = 20 \) in the series solution and those from the analytical solution (Eq. (28b)) ensure the accuracy of the series solution in Eq. (18). Similarly, for exponential and power-law variations of \( E(r) \) and \( v(r) \) given by Eq. (3b), the closeness of values of \( K_4 \) listed in Table 3 for different values of \( \gamma_1 \) computed using \( M = 10 \) in the series solution and those from the analytical solution (Eq. (28b)) verify the accuracy of the series solution in Eq. (21). Thus values of \( M \) to get a converged solution depend upon values of \( n, \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \).

5.2. For field uniaxial tensile loading

5.2.1. Stress concentration factors

For the uniaxial loading, values of \( K \) for different values of \( \beta_1, \beta_2 \), and \( n \) in Eq. (3a) listed below in Table 4 reveal that

(i) for \( \beta_1 > 0, \beta_2 < 0 \) the K for the FGM panel is larger (smaller) than that for the homogeneous material panel,

(ii) when the material parameter monotonically increases \( (\beta_1 < 0) \) with the distance from the hole center, the presently computed reduction in \( K \) for the FGM panel agrees with that reported in [19],

(iii) the maximum (minimum) value of \( K \), equals 4.83 (0.39) that is 1.61 (0.13) times the value of \( K \) for a homogeneous material.

Thus, the stress concentration factor can be reduced by a factor of about \( 8 = 3/0.39 \) by properly grading Young’s modulus and Poisson’s ratio in the radial direction.

For different values of \( \beta_1 \) and \( \beta_2 \), the variation of \( K \) with \( n \) is shown in Fig. 4. Here we have tacitly assumed that \( K \) continuously depends upon \( n \) since the analytical solutions has been found only for integer values of \( n \). It is seen from Fig. 4 that \( K \) increases (decreases) with an increase of the absolute value of \( n \) for \( \beta_1 > 0, \beta_2 < 0 \). From values of \( K \) listed in Table 4 and plotted in Fig. 4, we conclude that the value of \( \beta_1, \beta_2 \) that affects the radial variation of Poisson’s ratio (Young’s modulus) has very little (significant) influence on \( K \).

Table 5

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<thead>
<tr>
<th>( \gamma_1 )</th>
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<td>5.0</td>
<td>4.86841</td>
</tr>
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</table>

Table 6a

| \( \beta_1 = 0.9n = -1 \) | \( \gamma_1 = 0.64 \) | \( \beta_1 = 0.9n = -2 \) | \( \beta_1 = 0.9n = -3 \) | \( \beta_1 = 0.9n = -4 \) | \( \beta_1 = 0.9n = -5 \) |
|\( \frac{dE(\rho)}{d\rho} \) | -0.90 | -1.22 | -1.80 | -2.70 | -3.60 | -4.50 |
| \( K \) | 4.2260 | 4.2617 | 4.4516 | 4.6160 | 4.7286 | 4.8135 |
In order to better show the influence of the variation of $E$ on $K$, we set $\beta_2 = 0$ and find a least-squares fit to obtain the following relationship among $K$, $n$, and $\beta_1$.

$$K = 2.944 + 1.907\beta_1 - 0.023n - 0.581\beta_1^2 - 0.001n^2 - 0.104\beta_1 n$$

while $\beta_2 = 0$

(30)

The variation of $K$, on the $\beta_1 n$-plane, is displayed in Fig. 5. One can find values of these two variables for a desired value of $K$, and hence design the FGM panel.

For $E$ and $v$ varying according to Eq. (3b), values of $K$ for different values of $\gamma_1$ and $\gamma_2$ are listed in Table 5. We note that for $\gamma_1 > 0$ ($\gamma_1 < 0$) $K$ for a FGM panel is larger (smaller) than that for a homogeneous material panel. The maximum (minimum) value of $K$, equals 4.88 (1.75) that is 1.63 (0.58) times the value of $K$ for a homogeneous material panel. Values of $\gamma_1$ in the range $[-0.9, 0.9]$ have a negligible effect on $K$.

Based on the data in Table 5, we also find a least-squares fit to obtain the relationship between $K$ and $\gamma_1$ as

$$K = 3.000 + 1.704\gamma_1 + 0.385\gamma_1^2 + 0.034\gamma_1^3$$

while $\gamma_2 = 0$

(31)

For a given value of $K$, the nonlinear algebraic Eq. (31) can be iteratively solved for $\gamma_1$.

5.2.2. Dependence of $K$ upon $E(\varphi)|_{\varphi=1}$ and $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}

We now investigate the dependence of $K$ upon $E(\varphi)|_{\varphi=1}$ and $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}$. We set $v = 0.3$, and list values of $K$ in Tables 6a and 6b and 7a and 7b for two arbitrarily chosen values of $E(\varphi)|_{\varphi=1}$ and $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}$.

We conclude from values of $K$ listed in Tables 6a and 6b that $K$ slowly decreases with an increase in $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}$ for both values of $E(\varphi)|_{\varphi=1}$ and it is less than that of a homogeneous material when $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1} > 0$.

That is, the signs of $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}$ are opposite for simultaneously increasing values of $K$ and $E_0$. Values of $K$ listed in Tables 7a and 7b suggest that $K$ increases gradually with an increase in the value of $E(\varphi)|_{\varphi=1}$ for the two values of $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1}$. Based on these observations, we recommend that $\frac{dE(\varphi)}{d\varphi}|_{\varphi=1} > 0$ for FGM panels.

5.2.3. Stress distributions

For the FGM material defined by Eq. (3a), we have plotted in Fig. 6(a)-(c) the variation of the non-dimensional stresses along a radial direction for specific values of the angular coordinate, $\theta$, and the hoop stress vs. $\theta$ for $\rho = 1$.

It is clear that on the line $\theta = \pi/2$, for positive (negative) values of $\beta_1$ and $\beta_2$, the hoop stress is maximum (minimum) on the hole surface. However, when $\beta_1$ and $\beta_2$ have opposite signs, then the sign of $\beta_1$ in the expression for Young’s modulus determines the value of the hoop stress. For $\beta_1 = -0.9$, the hoop stress is nearly uniform on the hole surface. The decay of the hoop stress with $\rho$ near the hole in the FGM panel is steeper than that in the homogeneous material panel. Stresses are essentially constant for $\rho > 5$.

Even though the radial variation of Poisson’s ratio has very little effect on the stress distributions and hence the stress concentration factor, as demonstrated in [35,36], it noticeably affects the displacement field.

Table 6b

<table>
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<tr>
<th>$\gamma_1$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.92</td>
<td>-0.68</td>
<td>-1.0</td>
<td>-1.2</td>
<td>-1.0</td>
<td>-0.8</td>
</tr>
<tr>
<td>0.37</td>
<td>0.60</td>
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<td>1.3961</td>
</tr>
</tbody>
</table>

5.2.4. Design of FGM Materials

We have analytically found the stress concentration factor and the stress distributions in a panel with a circular hole and made of an
isotropic functionally graded material. Young’s modulus, $E$, and Poisson’s ratio are assumed to vary only in the radial direction either as general power-law functions or the former exponentially and the latter as a power law. Variation of $E$ governs stresses near the hole periphery and their rate of decay along a radial line. The stress concentration factor, $K$, can be decreased by a factor of 8 by suitably grading the material properties. Values of $K$ slowly decrease with an increase of the value of $\frac{dE}{d\rho}$, and it is less than that for a homogeneous material when $\frac{dE}{d\rho} > 0$. Furthermore, $K$ increases gradually with an increase in the value of $E(\rho)_{\rho=1}$.

These results suggest that one can tailor the radial variation of $E$ to achieve a pre-specified value of $K$, and essentially a uniform hoop stress in the entire panel. For the material problem an explicit expression for $E(\rho)$ is given for remote biaxial loading of a panel.

For remote uniaxial tensile loading, we have synthesized the computed results to express $K$ in terms of the material inhomogeneity parameters. One can use these relations to find values of the inhomogeneity parameters for a desired value of $K$.

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Appendix A

Expressions for $B_i(s)$ ($i = 1, 2, 3$) appearing in Eqs. (17a) and (17b) are given below.

$B_1(s) = (12n–32–8n^2–5n^3–n^4 + 8 + 3n^3–2s^4 + 4n^2\gamma_1–4n\gamma_2)\beta_1 + (2n\gamma_3–n^2\gamma_4–2n\gamma_5)n\beta_1 + (4n\gamma_6–4n\gamma_7 + n^2\gamma_8–2n\gamma_9 + n^2\gamma_{10})\beta_2$

$B_2(s) = (12n–16n–6n^2 + 4k^2n^2 + k^4n^4 + 4k^2n^4–k^4n^4 + 2k^2n^4–k^4n^4 + 16)\beta_1^2$

$+ (8kn^2–6n^2–10kn^2 + 12kn^2–2kn^2 + 6kn^2–4kn^2 + 8kn^2–12kn^2 + 16)\beta_2^2$

$+ (6kn^2–5n^2–8kn^2–6kn^2 + 4s^2 + 2n^3–4kn^2–4n\gamma_4 + 2n\gamma_4)n\beta_1^2 + (2n\gamma_5–n\gamma_6–k^2n\gamma_7 + n^2\gamma_8–2kn\gamma_9–n^2\gamma_{10})\beta_2^2$

$+ (4n^2 + 4n\gamma_6 + n\gamma_7)\beta_1\beta_2 + (k^2\gamma_8–kn\gamma_9–kn\gamma_10–2kn\gamma_9–n^2\gamma_{10} + 2kn\gamma_{10} + n^2\gamma_{11})\beta_2^2$

$B_3(s) = (2n^2–20n–32kn + 10kn^2 + 8kn^2 + n^2 + 13kn^2 + 19kn^2 + 8k^2n^2 + n^2–32)\beta_1$

$+ (10ns + 16kn–5k^2n^2–6kn^4–3k^2n^4 + 13kn^2 + 38kn^2 + 24kn^2–10kn^2)\beta_1$

$+ (8kn^2–8kn^2–8kn^2–8kn^2–8kn^2–18kn^2–12kn^2–8s^2\beta_1–(6n^3 + 8kn^2 + 2s)\beta_1)$

$+ (6n^2–4n + 2kn^2–3kn–k^2n^2 + 2n^3–3ns)n\gamma_1\beta_1–(2kn + sn)\gamma_2\beta_1 + (4–6n–2kn + 2n^2 + 3kn^2 + k^2n^2–2s + 3ns + s^2 + 2kn)\gamma_2\beta_2$

Appendix B

Expressions for $D_i(s)$ ($i = 1, 2, ..., 6$) appearing in Eqs. (20a)–(20c) are given below.

$D_1(s) = (–2s^3 + 8\gamma_1–12\gamma_1 + s(5–3\gamma_2)\gamma_1 + s^2(\gamma_3 + 6)\gamma_1–(5^2–3s + 8)\gamma_2\gamma_2$

$D_2(s) = (\gamma_2–4s^2 + s + s^2–2s + 16)\gamma_2\gamma_2$

$D_3(s) = (–2s^3 + 12\gamma_1–9)\gamma_1 + s(12 + \gamma_2)\gamma_1–s(5\gamma_3 + 13)\gamma_1–(5^2–5s + 12)\gamma_2$ ...

$D_6(s) = (s–k–4)\gamma_2\gamma_2$

$D_7(s) = (s^2 + k^2–5n^2 + 2)\gamma_2\gamma_2 + s(\gamma_3 + 3)\gamma_2–k(\gamma_2–2s + 3)\gamma_2 + k(7–2s)\gamma_2\gamma_1\gamma_2 + (k^2 + s^2–2s + 22)\gamma_2\gamma_2$

$D_8(s) = (2k^2–2s^3)\gamma_1 + 18(\gamma_1 + 1)\gamma_1 + s^2(18 + \gamma_2)\gamma_1 + k(18 + \gamma_2–6s)\gamma_1–2sk(18 + \gamma_2)\gamma_1–s(7\gamma_3 + 43)\gamma_1 + k(6s^2 + 7\gamma_2 + 43)\gamma_1–(k^2 + s^2–2s + 18)\gamma_2$

$–k(7–2s)\gamma_2$
References


