Micromechanical progressive damage analysis of inter- and intra-layer failures in fiber-reinforced composite laminates

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Abstract
We analyze progressive damage and failure of composite laminates by using a micromechanical bridging model and compare numerical and experimental results for three fracture problems, namely, the four-point bending, the simple tension, and the simple tension of a laminate with an open hole at its centroid. These problems involve fiber–matrix interface debonding, constituents' damage, interlayer delamination, and localized damage due to stress concentration. Macroscopic constitutive equations of unidirectional lamina, derived from those of the fiber and the matrix by using the bridging model with the fiber material assumed to be linearly elastic and the matrix to be elasto-plastic obeying the Drucker–Prager yield criterion, are employed. Strains in each constituent of the composite are assumed to be infinitesimal for the additive decomposition of strains into elastic and plastic parts to be valid, and the incremental plasticity theory is used. Stresses in the two constituents are found from their values in the homogenized material by using a dehomogenization technique. The intra-layer damage is assumed to initiate at a material point when the failure criterion for either the fiber or the matrix is satisfied. Young's modulus of a constituent is degraded by following a Weibull distribution. A finite element is deleted when an energy-based failure criterion is satisfied in it, and the analysis is continued till the structure fails. The delamination between adjacent plies is simulated by including a thin resin layer at the interface and studying failure initiation and propagation in it. The computed reaction force versus the displacement curves and the failure patterns in the three problems are found to agree with the corresponding experimental data.

Keywords
Constituent level failure, progressive damage analysis, ultimate failure, element deletion, homogenization and dehomogenization

Introduction
Carbon fiber-reinforced polymer (CFRP) composites are widely used in aerospace and shipbuilding industry due to their high specific mechanical properties and our ability to tailor their strengths in different directions. Thus, effective and reliable progressive failure analysis (PFA) methods for predicting the complex damage and fracture modes in CFRP composite structures are needed. The PFA, proposed by Kachanov,\textsuperscript{1} has been adopted in material/constitutive models to delineate damage initiation, damage propagation, material properties degradation, and ultimate failure of laminated and woven composites.\textsuperscript{2–24} Various proposed failure criteria include the maximum stress/strain, Tsai-Wu,\textsuperscript{25} Tsai-Hill,\textsuperscript{26} Hashin,\textsuperscript{27,28} Puck and Schürrmann,\textsuperscript{29} Huang and Lee,\textsuperscript{30} Linde et al.,\textsuperscript{31} and Hassan and Batra.\textsuperscript{32} Matzenmiller et al.’s\textsuperscript{2} continuum damage model is often used to monitor progressive damage evolution (PDE) in composites. As pointed out by Hassan and Batra, Coleman and Noll’s\textsuperscript{33} theory of internal variables has been employed by various authors, including

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Kachanov, with an internal variable considered as a damage variable. Shokrieh and Lessard\textsuperscript{7} proposed a “sudden degrade” method for ply moduli using material direction and damage-mode dependent factors that have been employed in Zhang et al.\textsuperscript{8} and McCarthy et al.\textsuperscript{9} Based on a micromechanical damage theory, Zhang et al.\textsuperscript{34} studied degradation of material properties of a unidirectional (UD) composite for different microscopic damages in the fiber and/or the matrix and established moduli degradation factors. We note that the ply-discounting technique may not preserve the symmetry of the degraded constitutive matrix\textsuperscript{10} for a symmetric ply laminate since a ply may fail due to either compressive or tensile stresses exceeding their limiting values.

Besides experimentally determining mechanical properties of an UD composite ply, various micromechanical theories have been effectively used to derive the composite moduli from those of the constituents. These theories include Hill’s equivalence energy principle,\textsuperscript{35} Palay and Aboudi’s cell model,\textsuperscript{36} Aghdam et al.’s simplified unit cell,\textsuperscript{37} Mori–Tanaka’s method,\textsuperscript{38} Hopkins and Chamis’ model,\textsuperscript{39} Eshelby’s equivalent inclusion method,\textsuperscript{40} Robertson and Mall’s free shear traction (FST) method,\textsuperscript{41} Huang’s bridging model (HBM),\textsuperscript{42} Mayes and Hansen’s multi-continuum theory,\textsuperscript{43} the rule of mixture (RoM), and the mechanics of materials approach. The predictions from Hopkins and Chamis’ model,\textsuperscript{39} the HBM,\textsuperscript{42} and Mayes’ multi-continuum theory\textsuperscript{43} are compared in the World Wide Failure Exercise.\textsuperscript{44} As listed in Table 4 of Hinton et al.,\textsuperscript{14} predictions from the HBM were found to agree with test findings.

During the last 20 years, the HBM has been improved upon to dehomogenize the overall stresses at a point of a lamina into in-situ stresses of constituents using stress concentration factors (SCFs)\textsuperscript{45,46} and analyze problems of three-phase materials\textsuperscript{47} for predicting failure of composites due to matrix cracking and fiber–matrix interface debonding.\textsuperscript{48} The FST and the HBM have, respectively, been employed by Batra et al.\textsuperscript{49} and by Nerilli and Vairo\textsuperscript{50} to study low velocity impact induced damage in laminates and progressive failure in composite bolted joints, and find stresses in fibers and the matrix from those in the homogenized material to characterize their failure. They used different failure criterion for each constituent, degraded moduli of the failed material, and deduced effective properties of the composite for subsequent use in damage propagation until ultimate failure of the structure. Other researchers have employed the HBM for studying deformations of composites.\textsuperscript{51–54} Here we employ the HBM and have assumed that strains are infinitesimal and incremental strains equal the sum of the elastic and plastic strain increments.

Many authors\textsuperscript{55–59} have used the finite element (FE) method coupled with a micromechanical theory to analyze failure in composites. For example, Batra et al.\textsuperscript{24} used the FST coupled with constituents’ failure criteria and degradation of their material parameters to analyze low velocity impact of FRPC laminates. Gutkin et al.\textsuperscript{56} studied the fiber compressive failure and kinking under in-plane shear and longitudinal compressive loading. Prabhakar and Waas\textsuperscript{57} explored the compressive strength and failure mode interaction in laminates using an upscale semi-homogenized model with 0° layers in the [45°/±45°/90°], laminate modeled with a micromechanical RVE and an hexagonal single fiber array. Huang Y et al.\textsuperscript{58} have developed the failure envelope of composite laminates by degrading constituents’ properties as a function of the damage induced in them. However, in view of the enormous literature on the subject, we may have missed some significant works.

Gopinath and Batra\textsuperscript{60} have recently provided a common framework for homogenizing material properties with Aboudi’s method of cells, the Fourier series analysis, and the transformation field analysis (TFA). They used these three micromechanics approaches to delineate the sensitivity of their predictions to the unit cell configuration taken as the cubic edge, the hexagonal close-packed, and the cubic-diagonal. The fiber cross-section was found to strongly affect both the elastic constants and the plastic hardening of the homogenized composite.\textsuperscript{61}

Here, we have implemented in the commercial FE software Abaqus/CAE, as a user defined subroutine VUMAT, the HBM that uses SCFs to find “true” stresses in the matrix.\textsuperscript{45,46} The VUMAT is used to homogenize material properties and find stresses at the constituent level, use the constituent failure criterion, degrade its material moduli, and deduce effective moduli of the damaged material for subsequent use. We have numerically studied the PDE in the four-point bending (FPB) of [0°/90°], laminate, simple tension (ST) of [0°/90°]_{4s}, [0°/±45°/90°]_{2s}, and [0°/±30°/±60°/90°]_{k} laminates, ST of [0°/90°]_{4s}, [0°/±45°/90°]_{2s}, and [0°/±30°/±60°/90°]_{k} laminates having an open hole tension (OHT) at the laminate centroid. We assume T300 fibers to be linearly elastic, and the 7901 epoxy to be an isotropic elasto-plastic material obeying the pressure-dependent Drucker–Prager yield criterion, the associated flow rule, and isotropic hardening. A fiber failure is identified by the maximum axial stress exceeding a critical value and the matrix failure by true stresses (stresses enhanced by the SCFs) in it satisfying the Tsai–Wu criterion with critical strength parameters found from the test data for the 7901 matrix specimens. Subsequent to a constituent’s failure, its modulus is degraded by using a Weibull distribution. For studying
delamination in FPB specimens, we add a thin resin layer between each pair of adjacent plies and analyze resin’s failure. In the OHT problem, stress concentrations are accounted for by parameters based on theoretical SCFs and the stress distribution around an open hole in an anisotropic lamina. A FE is deleted when the presumed failure criteria in it is satisfied. The analysis is continued until the entire structure collapses as indicated by a sudden drop in the total load in a displacement-controlled simulation. The computed load–displacement curve and the fracture patterns are found to agree with the corresponding test observations.

The rest of the paper is organized as follows. In the Section “Theoretical background,” we briefly describe the HBM and the Drucker–Prager yield criterion and derive the compliance matrix for infinitesimal elasto-plastic deformations. The progressive damage model including the failure criteria for the matrix, the fiber, the fiber/matrix debonding, and the delamination are summarized in the Section “Progressive damage model.” This section also describes simulation of delamination between adjacent plies by incorporating a thin layer of pure resin between them and studying resin’s failure. The experimental setups for the FPB problem and the simple tensile loading of a laminate with and without an open hole at its centroid are explained in the Section “Results.” This Section also details the computational model, the deduction of material properties from the test data, and a comparison of the computed and the test results for the three problems studied. Results presented include fringe plots of different failure indexes, and the failed FEs are deleted from the analysis. Conclusions of the work are summarized in the last section.

**Theoretical background**

*Homogenization and dehomogenization of material properties*

We define averaged stress and strain increments, \( \sigma_i \) and \( \varepsilon_i \), respectively, at a point of a UD composite as their volume averages over a RVE. That is

\[
\sigma_i = \int_{V'} \frac{d\sigma_i}{dV} dV' = V_f \sigma_{f,i} + V_m \sigma_{m,i} \quad (1a)
\]

\[
\varepsilon_i = \int_{V'} \frac{d\varepsilon_i}{dV} dV' = V_f \varepsilon_{f,i} + V_m \varepsilon_{m,i} \quad (1b)
\]

where \( V_f \) (\( V_m \)) is the volume fraction of the fiber (matrix) in the RVE, \( V' = V_f + V_m \) is the RVE volume, and \( \sigma_i = \{ \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{22}, \sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}, \sigma_{33}, \sigma_{34}, \sigma_{35}, \sigma_{36}, \sigma_{44}, \sigma_{45}, \sigma_{46}, \sigma_{55}, \sigma_{56}, \sigma_{66}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{22}, \sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}, \sigma_{33}, \sigma_{34}, \sigma_{35}, \sigma_{36}, \sigma_{44}, \sigma_{45}, \sigma_{46}, \sigma_{55}, \sigma_{56}, \sigma_{66} \} \). Recalling the mean-value theorem of integral calculus, equalities in equation 1(a) and (b) hold when \( \sigma_{f,i} \) and \( \sigma_{m,i} \) are mean values of stresses over the volumes \( V_f \) and \( V_m \), respectively. Similarly, \( \varepsilon_{f,i} \) and \( \varepsilon_{m,i} \) are average strain increments in the fiber and the matrix volumes, \( V_f \) and \( V_m \). Even though the mean-value theorem does not impose any restrictions on the magnitude of \( \varepsilon_{f,i} \) and \( \varepsilon_{m,i} \), strains are assumed to be infinitesimal for the additive decomposition of a strain increment into its elastic and plastic parts. Throughout the paper, we use superscripts and subscripts \( f \) and \( m \) to signify, respectively, quantities for the fiber and the matrix.

The constitutive relations of the fiber and the matrix relating their respective stress and strain increments are given by equation (1c-e) in which \( S \) is the compliance matrix

\[
\begin{align*}
\{ \varepsilon_i \} &= \left[ S_{ij} \right] \{ \sigma_i \} \quad (1c) \\
\{ \varepsilon_i \} &= \left[ S_{ij} \right] \{ \sigma_i \} \\
\{ \varepsilon_i \} &= \left[ S_{ij} \right] \{ \sigma_i \} \\
\{ \varepsilon_i \} &= \left[ S_{ij} \right] \{ \sigma_i \} \\
\{ \varepsilon_i \} &= \left[ S_{ij} \right] \{ \sigma_i \}
\end{align*}
\]

The micromechanical bridging model relates stress increments in the fiber and the matrix through the bridging matrix, \( A \), as

\[
\{ \sigma_i \} = \left[ A_{ij} \right] \{ \sigma_j \} \quad (2a)
\]

Combining equations 2(a) with 1(a) gives

\[
\begin{align*}
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} = \left[ B_{ij} \right] \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \}
\end{align*}
\]

where \( [I] \) is a unit matrix. Using equations 2(b), 2(c), 1(b), and 1(c) to (e), we get

\[
\begin{align*}
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \} \\
\{ \sigma_i \} &= \left( V_f [I] + V_m [A_{ij}] \right)^{-1} \{ \sigma_j \}
\end{align*}
\]

The 3D bridging matrix for a UD composite considering the matrix to be an elasto-plastic material is expressed as

\[
A_{ij} = \begin{bmatrix}
11 & 12 & 13 & 14 & 15 & 16 \\
0 & 22 & 23 & 24 & 25 & 26 \\
0 & 0 & 33 & 34 & 35 & 36 \\
0 & 0 & 0 & 44 & 45 & 46 \\
0 & 0 & 0 & 0 & 55 & 56 \\
0 & 0 & 0 & 0 & 0 & 66
\end{bmatrix}
\]
in which the off-diagonal elements are determined by substituting from equation 3(a) into equation 2(d) and by requiring the resulting compliance matrix to be symmetric. The non-zero independent elements of the bridging matrix, \([A_{ij}]\), are listed below with details provided in Huang ZM.65–67

\[
a_{11} = \frac{E_m}{E_{11}^f}, \quad a_{22} = a_{33} = a_{44} = 0.3 + 0.7 \frac{E_m}{E_{22}^f}
\]

(3b, c)

\[
a_{55} = a_{66} = 0.3 + 0.7 \frac{G_m}{G_{12}^f} \quad E_m = \begin{cases} E_m, & \text{if } \bar{\sigma}_v^m \leq \sigma_y^m \\ E_f, & \text{if } \bar{\sigma}_v^m > \sigma_y^m \end{cases}
\]

(3d)

\[
G_m = \begin{cases} 0.5 \frac{E_m}{1 + \nu_m}, & \text{if } \bar{\sigma}_c^m \leq \sigma_y^m \\ \frac{E_m}{3}, & \text{if } \bar{\sigma}_c^m > \sigma_y^m \end{cases}
\]

(3e)

In equation (3) \(E_{ij}^f\), \(v_{ij}^f\), \(G_{ij}^f\) and \(E_m\), \(v_m\), \(G_m\) are Young’s moduli, Poisson’s ratios, and the shear moduli of the fiber (assumed to be transversely isotropic with the axis of transverse isotropy along the fiber) and the matrix (assumed to be isotropic), respectively. \(\sigma_y^m\) and \(\sigma_y^m\) are, respectively, the yield stress and the effective true stress of the matrix, and \(E_{ij}^m\) is a tangent modulus of the matrix in the plastic range. The elastic parameters of the fiber and the elasto-plastic parameters for the matrix are, respectively, listed in Tables 1, 2, and 3. The moduli of the homogenized composite are listed in Appendix B.

**Drucker–Prager yield criterion**

Uniaxial compressive and tensile tests on the 7901 epoxy gave different yield stresses.68 We postulate that the pressure-dependent Drucker–Prager yield surface provides a reasonable representation of its response. Defining the effective stress as

\[
\bar{\sigma} = \sqrt{3F(\sigma_{ij})} + \alpha^m \sigma_{kk}
\]

(4a)

where \(\alpha^m\) accounts for the hydrostatic pressure effect; a repeated index implies summation over its range, 1, 2, 3, and

\[
F(\sigma_{ij}) = \frac{1}{6} \left[ (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + (\sigma_{11} - \sigma_{22})^2 \right] + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2
\]

(4b)

the yield surface is described by

\[
f \equiv \sqrt{3F(\sigma_{ij})} + \alpha^m \sigma_{kk} - \beta^m = 0
\]

(4c)

With \(\sigma_y^f\) and \(\sigma_y^m\), respectively, equaling the axial yield stress in uniaxial tensile and compressive deformations of the epoxy, we have

\[
\sigma_y^f + \alpha^m \sigma_y^m = \sigma_y^m - \alpha^m \sigma_y^m = \beta^m
\]

(4d)

These equations relate \(\alpha^m\) and \(\beta^m\). Batra et al.49 have proposed taking \(\sigma_y^f\) and \(\sigma_y^m\) as the proof stresses corresponding to 0.2% effective plastic strain.

In any incremental deformation, the effective plastic strain increment, \(d\bar{\varepsilon}\), satisfies

\[
d\bar{\varepsilon}_{ij} = \bar{\sigma} d\bar{\varepsilon}
\]

(4e)

We use the following flow rule

\[
d\bar{\varepsilon}_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}
\]

(4f)

where \(d\lambda\) is the proportionality factor. Note that \(d(\lambda) = 0\) for \(f < 0\) in equation (4c).

Substituting for \(d\bar{\varepsilon}\) from equation (4f) into equation 4(e) and making use of equation 4(a), we obtain

\[
d\lambda = d\bar{\varepsilon} = \frac{d\bar{\sigma}}{H^m}
\]

(4g)

where \(H^m(\bar{\varepsilon}) = \frac{d\sigma}{d\varepsilon}\) is the strain-hardening modulus and is usually found by numerically differentiating the

| Table 2. 7901 Epoxy’s elastic properties and ultimate strengths.66 |
|-------------------|---------------|------------------|------------------|
| \(E^m/\text{GPa}\) | \(G^m/\text{GPa}\) | \(\nu^m\) | \(\sigma_{uu}^m/\text{MPa}\) | \(\sigma_{ud}^m/\text{MPa}\) | \(\sigma_{ud}^m/\text{MPa}\) |
| 3.17              | 1.02          | 0.355           | 85.1             | 52.6             | 106.4            |

| Table 3. 7901 Epoxy’s strain-hardening and the Drucker–Prager yield criterion parameters. |
|-------------------|---------------|------------------|
| \(\alpha^m\) | \(\beta^m\) | \(A/\text{MPa}\) | \(N/\text{MPa}\) |
| 0.238             | 49.52         | 127.095          | 10.166          |

| Table 1. T300 fiber’s elastic properties and ultimate strengths.86 |
|-------------------|---------------|------------------|
| \(E_{11}^f/\text{GPa}\) | \(E_{22}^f/\text{GPa}\) | \(G_{12}^f/\text{GPa}\) | \(G_{23}^f/\text{GPa}\) | \(\nu_{12}\) | \(\nu_{23}\) | \(\sigma_{uu}^e/\text{MPa}\) | \(\sigma_{ud}^e/\text{MPa}\) |
| 230               | 15            | 15               | 7                | 0.2            | 0.2            | 2500             | 2000             |
effective stress versus the effective plastic strain curve. Alternatively, one can fit by the least squares method a function
\[
\bar{\varepsilon} = \exp\left(\frac{\bar{\sigma} - A}{N}\right)
\]  
(4h)
to the test data.\(^3\) Then
\[
H^m(\bar{\varepsilon}) = \frac{d\bar{\sigma}}{de^p} = \frac{d(A + N\ln(\bar{\varepsilon}))}{de^p} = \frac{N}{\bar{\varepsilon}} = \frac{N}{e^\frac{2\bar{\varepsilon}}{N}}
\]
(4i)
The differential of both sides of equation 4(a) can be written as
\[
d\bar{\sigma} = \left[\frac{3}{2(\sigma - \alpha^m\sigma_{kk})}(\sigma_{ij} - \alpha^m\delta_{ij}) + \alpha^m\delta_{ij}\right]d\sigma_{ij}
\]
(4j)
Substituting for \(d\bar{\sigma}\) from equation 4(g) and for \(d\bar{\varepsilon}\) from equation 4(j) into equation 4(f) gives
\[
de_{ij}^p = \frac{1}{H^m} \left[\frac{3}{2(\sigma - \alpha^m\sigma_{kk})}(\sigma_{ij} - \alpha^m\delta_{ij}) + \alpha^m\delta_{ij}\right]d\sigma_{ij}
\times \left[\frac{3}{2(\sigma - \alpha^m\sigma_{kk})}(\sigma_{pq} - \alpha^m\delta_{pq}) + \alpha^m\delta_{pq}\right]d\sigma_{pq}
\]
\[= S_{ijkl}^{m}\bar{\varepsilon}\bar{\varepsilon}\]
(4k)
where \(S_{ijkl}^{m}\) is the elasto-plastic compliance matrix of the epoxy.

For a uniaxial tension test, the effective stress \(\bar{\sigma}\) and the effective plastic strain \(\bar{\varepsilon}\) can be expressed in terms of the axial stress \(\sigma_{11}\) and the axial plastic strain \(\varepsilon_{11}^p\) as
\[
\bar{\sigma} = (1 + \alpha^m)\sigma_{11}\quad \text{and} \quad \bar{\varepsilon} = \frac{\varepsilon_{11}^p}{1 + \alpha^m} = \frac{\varepsilon_{11} - \sigma_{11}/E^m}{1 + \alpha^m}
\]
(4l, m)

The analogous relation in a uniaxial compression test is
\[
\bar{\sigma} = (1 - \alpha^m)\sigma_{11}\quad \text{and} \quad \bar{\varepsilon} = \frac{|\varepsilon_{11}^p|}{1 - \alpha^m} = \frac{|\varepsilon_{11} - |\sigma_{11}|/E^m}{1 - \alpha^m}
\]
(4n, o)

For infinitesimal elasto-plastic deformations, we get
\[
de = de^p + de^\varepsilon = S_{ijkl}^{m}\bar{\varepsilon}\bar{\varepsilon}d\sigma + S_{ijkl}^{m}\bar{\varepsilon}d\varepsilon
\]
(4p, q)

### Progressive damage model

#### Intra-layer damage and failure analysis

**Matrix failure.** We use the Tsai–Wu criterion for ascertaining the matrix failure (equation (5a)) at a point in an UD composite. Huang and Xin\(^b\) have pointed out that the transverse, the axial, and the shear stresses in a fiber differently influence a matrix failure. In the Tsai–Wu criterion given by equation 5(a), all six stress components are considered.

\[
P_m = F_{11}(\bar{\sigma}_{11}^{m})^2 + F_{22}(\bar{\sigma}_{22}^{m})^2 + F_{33}(\bar{\sigma}_{33}^{m})^2 + F_{12}(\bar{\sigma}_{12}^{m})^2 + F_{23}(\bar{\sigma}_{23}^{m})^2 + F_{13}(\bar{\sigma}_{13}^{m})^2
\]
(5a)
where
\[
F_{ij} = \frac{1}{\sigma_{ij}^{m,n, t, c}} - \frac{1}{\sigma_{ij}^{m,n, u, t, c}}
\]
(5b–e)
\(\sigma_{ij}^{m,n, t, c}\), \(\sigma_{ij}^{m,n, u, t, c}\), and \(\sigma_{ij}^{m}\) are the ultimate tensile, compressive, and shear strengths of the matrix. \(\bar{\sigma}_{ij}\) is the true stress component of the matrix, which is computed from the incremental homogenized stress \(\bar{\sigma}_{ij}^{m}\) using equation 2(c) and the appropriate SCFs. The true stresses are defined as
\[
(\bar{\sigma}_{11}^{m})_i = (\bar{\sigma}_{11}^{m})_{i-1} + d\sigma_{11}^m
\]
(5f)
\[
(\bar{\sigma}_{22}^{m})_i = (\bar{\sigma}_{22}^{m})_{i-1} + K_{22}d\sigma_{22}^m
\]
(5g)
\[
(\bar{\sigma}_{33}^{m})_i = (\bar{\sigma}_{33}^{m})_{i-1} + K_{33}d\sigma_{33}^m
\]
(5h)
\[
(\bar{\sigma}_{12}^{m})_i = (\bar{\sigma}_{12}^{m})_{i-1} + K_{12}d\sigma_{12}^m
\]
(5i)
\[
(\bar{\sigma}_{13}^{m})_i = (\bar{\sigma}_{13}^{m})_{i-1} + K_{13}d\sigma_{13}^m
\]
(5j)
\[
(\bar{\sigma}_{23}^{m})_i = (\bar{\sigma}_{23}^{m})_{i-1} + K_{23}d\sigma_{23}^m
\]
(5k)

\(K_{ij}^m\) and \(\hat{K}_{ij}^m\) are the transverse tensile SCFs of the matrix before and after the interface debonding, \(K_{22}^m\) and \(K_{23}^m\) are, respectively, the matrix transverse compressive and shear SCFs, \(K_{12}^m\) and \(K_{13}^m\) are, respectively, the in-plane shear SCFs, and \(K_{13} = K_{12}\). Since the UD
composite is taken to be transversely isotropic, $K_{33} = K_{22}$ \cite{47}. Details of the SCFs are provided in Wang and Huang \cite{47} and Zhao YQ et al., \cite{68} and their formulae are summarized in Appendix 1.

We postulate that the fiber–matrix interface debonding occurs when \cite{68}

$$P_d = \sqrt{(\sigma_{ij}^m)^2 + (K_{22}^m)^2 - K_{22}^m \sigma_{ij}^m + 3(K_{12}^m)^2/\sigma_{ij}^r} \geq 1$$

(5m)

where $\sigma_{ij}^m$ is the current homogenized stress in the matrix, and $\sigma_{ij}^m$ is the critical von Mises stress for detecting the fiber–matrix interface debonding. Values of SCFs are listed in Table 4.

Fiber failure. The fiber is assumed to fail when the maximum axial stress in it exceeds the critical value. That is

$$|P_f| \geq 1, \quad P_f = \frac{\sigma_{ij}^f}{\sigma_{ij}^u} \left( \begin{array}{l} \sigma_{ij}^f = \sigma_{ij}^u, \text{if} \quad \sigma_{ij}^f \geq 0 \\ \sigma_{ij}^f = \sigma_{ij}^u, \text{if} \quad \sigma_{ij}^f \leq 0 \end{array} \right)$$

(6a)

where $\sigma_{ij}^f$ = $\sigma_{ij}^u$ + $\Delta \sigma_{ij}^f$ and $\Delta \sigma_{ij}^f$ is calculated from equation 2(b). The ultimate tensile and compressive strengths of the fiber, $\sigma_{ij}^u$ and $\sigma_{ij}^c$, respectively, are listed in Table 1.

Post failure analysis. Recall that when either the fiber or the matrix fails at a material point, the structure still has considerable load carrying capacity. We use the Weibull distribution to gradually degrade the elastic modulus \cite{10} and the stiffness parameters \cite{69} of the FE in which a constituent has failed. The Weibull degradation is expressed in terms of its failure index $P_m$ or $P_f$. Thus

$$(E_{ij}^d)^d = (1 - w_f)E_{ij}, \quad (G_{ij}^d)^d = (1 - w_f)G_{ij}$$

$$w_f = 1 - \exp \left( -\frac{(P_f - 1)^{m_f}}{m_f} \right)$$

(7a-c)

$$(E_{ij}^d)^d = (1 - w_m)E_{ij}, \quad (G_{ij}^d)^d = (1 - w_m)G_{m}$$

$$w_m = 1 - \exp \left( -\frac{(P_m - 1)^{m_m}}{m_m} \right)$$

(7d-f)

Quantities with superscript “d” are for the damaged or degraded material, and $w_m$ and $w_f$ are, respectively.

Table 4. SCFs and critical von Mises stresses of the 7901 epoxy. \cite{46}

<table>
<thead>
<tr>
<th>$K_{22}$</th>
<th>$K_{22}$</th>
<th>$K_{22}$</th>
<th>$K_{12}$</th>
<th>$K_{23}$</th>
<th>$\sigma_{ij}^c$/MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.37</td>
<td>5.54</td>
<td>1.81</td>
<td>1.17</td>
<td>2.29</td>
<td>35.89</td>
</tr>
</tbody>
</table>

Damage variables for the matrix and the fiber in the traditional sense. In order not to have singular strains at a material point, the maximum value of the parameter, $m_i$, in equation 7(c) and (f) is set equal to 0.8. A damaged material point is assumed not to heal. The degradation of an UD layer’s moduli due to constituents’ failure is illustrated in Appendix 2.

Ultimate failure criteria of a material point. An energy-based fracture criterion, widely used for fracture detection, \cite{20-23,70} is adopted to identify the final failure of the material in an FE. For the OHT problem, the mesh size effect is minimized by setting the characteristic length (CL) proposed by Whitney and Nuismer \cite{71} equal to the radial distance of the point from the hole boundary; e.g., see Chang et al., \cite{72} Ramkumar et al., \cite{73} Ramkumar et al., \cite{74} Eriksson et al., \cite{75} Whitworth et al., \cite{76} Camanho and Lambert. \cite{77} This did not completely eliminate the dependence of the numerical solution upon the mesh density. It is hard to specify a CL value without experimental data for a general laminate with holes of different radii. \cite{78-80} Accordingly, we use the following element deletion criteria.

For $0^\circ$ UD composite

$$|P_c| \geq 1, \quad P_c = \frac{\varepsilon_{11} \sigma_{11} d_{ij}^c}{2U_{ij} K_{Y}^c}$$

(8a)

For off-axis UD composite

$$|P_c| \geq 1, \quad P_c = \frac{\varepsilon_{11} \sigma_{11} d_{ij}^c + \varepsilon_{22} \sigma_{22} d_{ij}^c + \varepsilon_{12} \sigma_{12} d_{ij}^c}{2U_{ij} K_{Y}^c}$$

and

$$U_{22}^c = \begin{cases} U_{22}^c, & \text{if} \quad \sigma_{22} > 0 \\ U_{22}^c, & \text{if} \quad \sigma_{22} \leq 0 \end{cases}$$

(8b)

where $U_{ij}^c$ are the critical strain energy densities of different failure modes obtained by equating results of fracture tests and the corresponding FE analyses of the UD composite without holes. \cite{68} We note that the out-of-plane stresses are not included in equation 8(b). Thus for out-of-plane deformations, the fracture energy is tacitly assumed to be very large due to the constraining effect of the adjacent UD layers in the thickness direction. Values of $U_{ij}^c$ are listed in Table 5.

In equation 8(a) and (b), the attenuation coefficient $d_{ij}^c$ minimizes the peak stress differentials due to the FE mesh size effect in the OHT problem, and the superscript $\varphi$ equals the offset angle between the external load and the fiber axis. We found that the size of FEs abutting the hole periphery strongly influenced the ultimate failure of the entire laminate. For a UD
Table 5. Critical strain energy densities for different failure modes of the UD composite.

<table>
<thead>
<tr>
<th>Critical strain energy density (10⁶ J/m²)</th>
<th>Axial tensile and compressive failure</th>
<th>Transverse tensile failure</th>
<th>Transverse compressive failure</th>
<th>Shear failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_{f_{11}}$</td>
<td>12.06</td>
<td>0.134</td>
<td>0.447</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Composite with offset angle $\psi$, we define $d_i^\psi$ as

$$d_i^\psi = \frac{\sigma_i^\infty(a, 0)}{\sigma_i^\infty(a + l/2, 0)}$$  \hspace{1cm} (8c)

Here $\sigma_i^\infty(x, 0)$ is the normal stress along the $x$- (fiber-) direction when the transversely isotropic lamina has the applied transverse normal stress $\overline{\sigma}_y$, $\sigma_i^\infty(a, 0)$ is the normal stress at the open hole edge ($x = a$), $\sigma_i^\infty(a + l/2, 0)$ is the integrated stress at the central node of the element at the free edge of the open hole, and $l^2$ is the volume of the edge element. The stress distribution along the $x$-direction is calculated through equation 8(d) 62

$$\sigma_i^\infty(x, 0) = \overline{\sigma}_y + \overline{\sigma}_y Re\left\{ \frac{1}{\mu_1 - \mu_2} \right. \right.$$

$$\left. \times \left[ \frac{-\mu_3(1 - i\mu_4)}{\sqrt{\gamma^2 - 1 - \mu_2^2}(\gamma + \sqrt{\gamma^2 - 1 - \mu_2^2})} \right] \right.$$

$$\left. + \frac{\mu_1(1 - i\mu_2)}{\sqrt{\gamma^2 - 1 - \mu_2^2}(\gamma + \sqrt{\gamma^2 - 1 - \mu_2^2})} \right\}$$  \hspace{1cm} (8d)

where $\gamma = x/a$, and $\mu_1$ and $\mu_2$ are solutions of the characteristic equation

$$s_{21}^2 \mu^4 - 2s_{26} \mu^3 + (2s_{12} + s_{66}) \mu^2 - 2s_{16} \mu + s_{11} = 0$$  \hspace{1cm} (8e)

where $s_{ij}, i, j = 1', 2', 6'$ are the laminate compliances. Directions $1'$ and $2'$ are parallel and transverse to the loading direction, respectively. When the effect of plastic deformations can be neglected, equation 8(e) simplifies to

$$s_{21}^2 \mu^4 + (2s_{12} + s_{66}) \mu^2 + s_{11} = 0$$  \hspace{1cm} (8f)

For the 0° laminate we have

$$S_{11} = \frac{V_f}{E_{11}^f} + \frac{\alpha_{11} V_m}{E_{1m}}$$  \hspace{1cm} (8g)

It is noted that $a_{ij}, i, j = 1, 2, 6$ are elements of the bridging tensor given by equation 3(b) to (d) and

$$a_{12} = \left( S_{12} - S_{66}^0 \right) \left( \alpha_{11} - \alpha_{22} \right) \left( S_{12} - S_{66}^0 \right)^T$$

$$s_{12} = -v_{12}/E_{11}, \quad s_{66} = -v_{66}/E_m, \quad S_{12} = 1/E_{11}, \quad \text{and} \quad S_{66} = 1/E_m.$$ Only principal roots of equation 8(f), i.e., two of the four roots that have positive imaginary part, are used. For off-axis loaded laminate, we first transform the compliance matrix from the local (material principal) axes to the global axes using the following coordinate transformation matrix

$$[S_{ij}] = [T][S_{ij}][T]^T$$  \hspace{1cm} (8k)

$$T = \begin{bmatrix} \frac{\bar{p}_1}{m_{11}} & \frac{\bar{p}_2}{m_{12}} & \bar{l}_1 \bar{l}_2 \\ \bar{m}_{11} & \bar{m}_{12} & \bar{m}_{13} \\ \bar{l}_1 \bar{m}_{11} & \bar{l}_2 \bar{m}_{12} & \bar{l}_2 \bar{m}_{13} \end{bmatrix}$$  \hspace{1cm} (8l)

$$l_1 = m_2 = \cos\varphi, \quad l_2 = -m_1 = -\sin\varphi$$  \hspace{1cm} (8m, n)

For the 0° and the 90° plies in the laminate studied in this work, values of $d_i$ for typical values of $l$ are listed in Table 6.

Without knowing the critical length (CL) for the UD laminate used in the OHT tests, we use the theoretical maximum SCF around a circular hole in an infinitely wide plate. The $K_f^\infty$ is calculated from equation 8(o) to (r) 64

$$K_f^\infty = \frac{E_0}{E_{11}} \left( -\cos^2\varphi + (k + n)\sin^2\varphi \right) \cos^2\theta$$

$$+ \left( (1 + n) \cos^2\varphi - k \sin^2\varphi \right) \sin^2\theta$$

$$- n (1 + k + n) \sin\varphi \cos\varphi \sin\theta \cos\theta$$  \hspace{1cm} (8o)
By including a resin layer at the interface between two adjacent UD layers, Petrossian and Wisnom\textsuperscript{81} used the Prandtl-Reuss flow rule and the critical strain energy release rate criterion for the resin to study its yielding and delamination at the interface. Fletcher et al.\textsuperscript{82} used Camanho’s\textsuperscript{83} quadratic damage onset criterion for the resin layer to study the delamination. We simulate delamination between two adjacent UD plies by introducing between them a thin layer of pure resin and assuming the delamination to initiate at a point when the stress state in the resin there satisfies the following Marion’s failure criterion\textsuperscript{84}

\[
P_f = \frac{\sigma_{xx}^2 + \sigma_{yy}^2}{\sigma_{xx,t}} + \left(\frac{\sigma_{zz}^2 + \sigma_{zz,t}}{\sigma_{zz,t}}\right) \geq 1
\]  

For the 7901 epoxy, we take \(\sigma_{xx,t} = 85.1\) MPa and \(\sigma_{zz,t} = 52.6\) MPa.\textsuperscript{68} The linear term in \(\sigma_{zz,t}\) in equation 9(a) simulates a delay in the onset of delamination due to a compressive inter-layer normal stress. When the failure criterion (9) is satisfied in an FE, it is deleted leading to a void and stress concentration. The delamination propagates as additional FEs are deleted.

### Ultimate failure

In a displacement-controlled test, the ultimate strength of a laminate is based on the maximum load reached prior to the structure collapsing as indicated by a sharp drop in the load.

### Recapitulation of damage and failure variables

Variables \(P_f\) and \(P_m\), respectively, defined by equations 6(a) and 5(a)) are used in equation 7(a) and (d) to find damage variables \(w_f\) and \(w_m\) that degrade elastic moduli of the two materials. The variables \(P_d\) and \(P_s\), respectively, found from equations 5(m) and 9(a) are used to check debonding at the fiber–matrix interface and to simulate delamination between adjoining layers.

### Results

#### Experimental work

**Four-point bending.** Thirty-two 0.125-mm-thick UD carbon fiber (T300)-reinforced epoxy 7901 prepregs with 62 % fiber volume fraction \((V_f)\) were hand laid as fiber sequence of \([0^\circ/90^\circ]_s\), then cured for 2.5 h in an autoclave at 120°C and 750 kPa pressure. Specimens with nominal dimensions of 100 × 24 × 4 mm\(^3\) were carefully cut from the cured panels. FPB test was conducted on an Electromechanical Universal Testing Machine (WD-20 A), as shown in Figure 1. The specimens resting on two 80 mm apart lower supports having hemispherical heads of radius 1.5 mm was loaded by two 42 mm apart upper hemispherical nosed pistons of radius 4 mm and moving downwards at 1 mm/min. The specimen was symmetrically placed with respect to the upper and the lower supports.

**ST and open hole tension tests.** Specimens with length \(\times\) width of 200 × 20 mm for ST and OHT tests were hand laid from the UD T300/7901 prepregs with fiber sequence of \([0^\circ/90^\circ]_{4s}\), \([0^\circ/\pm 45^\circ/90^\circ]_s\), and \([0^\circ/\pm 30^\circ/\pm 60^\circ/90^\circ]_s\). After curing, they were cut as mentioned above. Two of the authors (YQZ and RCB) used individual smart phones to simultaneously record the cracking sounds emitted from a \([0^\circ/\pm 45^\circ/90^\circ]_s\) laminate during loading for verifying our earlier observed

---

### Table 6. Typical values of \(l\) and \(d_i\) for the 0° and the 90° UD layer.

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>15°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l) (mm)</td>
<td>0.319</td>
<td>0.239</td>
<td>0.159</td>
<td>0.0798</td>
<td>0.0398</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d_i) (90° fiber)</td>
<td>1.201</td>
<td>1.169</td>
<td>1.136</td>
<td>1.103</td>
<td>1.086</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d_i) (0° fiber)</td>
<td>2.727</td>
<td>2.475</td>
<td>2.014</td>
<td>1.599</td>
<td>1.363</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 7. Values of \(K_f^r\) around an open hole in an UD layer.

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>15°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>75°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_f^r)</td>
<td>7.57</td>
<td>7.16</td>
<td>6.04</td>
<td>4.50</td>
<td>2.95</td>
<td>2.56</td>
<td>2.64</td>
</tr>
</tbody>
</table>

\[
k = \frac{E_{11}}{E_{22}} \quad n = \sqrt{2 \left( \frac{E_{11}}{E_{22}} - \mu_1 \right) + \frac{E_{11}}{G_{12}}} \quad (8p, q)
\]

\[
E_\theta = \frac{1}{\sqrt{\left( \sin^4 \theta \frac{E_{11}}{E_{44}} + \left( \frac{1}{G_{12}} \frac{2 \mu_1}{E_{44}} \sin^2 \theta \cos^2 \theta + \frac{\cos^4 \theta}{E_{22}} \right) \right)}} \quad (8r)
\]

Here \(E_{11}, E_{22}, G_{12}, v_{12}\) are elastic moduli of the UD composite obtained from the HBM, and \(\theta\) is the polar angle with respect to the x-axis. \(K_f^r\) is a periodic function of angle \(\theta\), and \(K_f^r\) is the maximum value of \(K_f^r\).

For laminates with \(\varphi = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ,\) and \(90^\circ,\) values of \(K_f^r\) are listed in Table 7.

In the absence of stress concentration, \(\sigma_{xx}^r\) and \(K_f^r\) equal 1.0. A FE is deleted from the analysis domain when the ultimate failure criterion is satisfied in it.

### Inter-layer failure or delamination

By including a resin layer at the interface between two adjacent UD layers, Petrossian and Wisnom\textsuperscript{81} used the Prandtl-Reuss flow rule and the critical strain energy release rate criterion for the resin to study its yielding and delamination at the interface. Fletcher et al.\textsuperscript{82} used Camanho’s\textsuperscript{83} quadratic damage onset criterion for the resin layer to study the delamination. We simulate delamination between two adjacent UD plies by introducing between them a thin layer of pure resin and assuming the delamination to initiate at a point when the stress state in the resin there satisfies the following Marion’s failure criterion\textsuperscript{84}

\[
P_f = \frac{\sigma_{zz}^2}{\sigma_{xx,t}^2} + \left(\frac{\sigma_{zz}^2 + \sigma_{zz,t}}{\sigma_{zz,t}}\right)^2 \geq 1
\]
results. The noise in the recorded sound prevented us from deciphering precisely when either delamination or matrix/fiber cracking started. The OHT specimens had a 6-mm-diameter circular hole carefully machined at their center using a standard Tungsten-tip drill with a cooling liquid. Fiberglass composite tabs were fixed to the specimen ends to avoid damage outside the 100 mm gauge length. Following the ASTM D 5776-2002 standard, the tensile tests were conducted on a WD-20 A Testing Machine at a constant displacement rate of 1.25 mm/min at room temperature using the test setup shown in Figure 1.

Computational work

FE modeling using Abaqus. The theoretical formulation described in the Section “Progressive damage model” has been implemented in Abaqus/Explicit using a VUMAT and was verified by analyzing simple problems with known analytical solutions. It was subsequently used to analyze deformations for the following four loading cases that were also studied experimentally: the pure resin, the FPB of a laminate, the ST deformations during off-axis loading of an UD laminate, and the OHT of a laminate. The flow chart for numerically studying damage at a material point of a UD composite lamina is given in Figure 2.

In VUMAT, the calculated strain increments for each FE are transferred back to Abaqus. The stiffness tensor of the FE at the current load step is determined through the implemented material constitutive model and is based on the overall stresses at the previous load step. The stress increments are then calculated from the strain increments to update the true stresses. If a constituents’ failure criteria for the FE is fulfilled but does not correspond to the ultimate energy failure criteria as given in Eqs. (8a) and (8b), a post failure analysis is initiated, and its stiffness degraded until it’s ultimate failure when it is deleted creating a void.

The surface-to-surface contact algorithm in Abaqus is employed to avoid interpenetration between adjoining layers.

The analysis procedure for the pure resin layer is simpler than that described above. We first calculate the stress and the strain increments based on the implemented constitutive model, and then compute the failure index (state variable $P_r$) based on the overall stresses. The FE is deleted when $P_r = 1$.

Estimation of resin parameter values from the test data. The experimental axial stress versus the axial strain curves for specimens tested in uniaxial tension and compression are exhibited in Figure 3(a). By taking the proof-stresses for axial strain = 0.4% as the yield stress, as shown by the dotted lines in Figure 3(a), we found $\alpha^n = 0.238$ and $\beta^n = 49.52$ MPa appearing in equation 4(d). The test data for the uniaxial tension/compression and torsional deformations were converted to the effective stress vs. the effective plastic strain curves exhibited in Figure 3(b). The least squares fit to the data for these curves provided $A = 127.095$ MPa and $N = 10.166$ MPa for equation 4(h).

In Figure 3(b) we have compared the effective stress vs. the effective plastic strain curves found from the test data with those computed using Abaqus and values of parameters from Tables 2 and 3. We see that the Drucker–Prager yield surface with exponential hardening provides a reasonable fit to the test data.
Example problem 1: four-point bending (FPB) of $[0^\circ/90^\circ]_{8s}$ laminates. The FPB of $[0^\circ/90^\circ]_{8s}$ laminate was simulated in Abaqus/Explicit by including in the 32 UD plies 21 0.0114-mm-thick layers of pure resin between two adjoining 0.114-mm-thick plies except that the delamination was not allowed to occur in the upper and the lower five interfaces since the test setup prevented it. The specimen geometry identical to that tested and the FE discretization into 101,940 C3D8R (Abaqus terminology; eight-node brick elements with one-point integration rule for evaluating elemental matrices) are illustrated in Figure 4(a). As shown in Figure 4(a), the two rigid rollers applying vertically downward loads on the top surface and the two stationary rigid rollers supporting the bottom surface are, respectively, 42 mm and 80 mm apart. Each ply and the resin layer was meshed with 20 FEs along the width and 80 FEs along the length. The FE mesh in areas of high probability of damage is finer than that in other areas. The material principal directions were prescribed in each layer.

The specimen was loaded by linearly increasing the vertical displacement of the upper two rollers. In the explicit algorithm, we used the mass scaling factor of 1000 to reduce the CPU time and limit the total kinetic energy of the specimen to a small value. In order to restrain stress concentration near the supports, a “softened” contact relationship was used, i.e., the contact surfaces transmitted pressure when the overclosure between them, measured in the contact (normal) direction, is positive. The slope, 9.285 GPa, of the pressure-overclosure relation was set equal to the elastic modulus in the normal direction of the contacting lamina layer. Damage modes are illustrated through fringe plots of the state variables, SDV7 = $P_f$, SDV8 = $P_m$, SDV15 = $P_{d}$, and SDV17 = $P_e$, which, respectively, denote failure indexes of the fiber, the matrix, the fiber–matrix interface, and the final failure of an FE within a lamina layer. For the pure resin layer, SDV8 = $P_e$ denotes the failure index of the pure resin material.
As shown in Figure 4(b) and (c), either discretizing each ply into more elements or reducing the thickness of the pure resin layer between two plies considerably increased the CPU time but introduced minute differences in the value of the dominant stress components, see Figure 4(d) to (f). For the remainder of the work, each ply and the pure resin layer are meshed with one FE through the thickness.

As shown in Figure 4(g), during the quasi-static FPB of \([0/90]_8s\) laminates, the energy balance is well satisfied.

Example problem 2: Simple tension of \([0/90]_4s, [0/\pm 45/90]_2s\), and \([0/\pm 30/\pm 60/90]_2s\) laminates.

In Abaqus/Explicit, the ST configuration with the same geometry as in the experiment is deformed by applying to the specimen end faces the axial displacement, \(u_x = 2 \text{ mm}\), along the global \(x\)-axis, as shown in Figure 5(a). Due to symmetry in the specimen geometry, boundary conditions, and the loading, deformations of only one-half of the specimen are analyzed by setting \(u_z = 0\) at points on the laminate mid-surface. Once stresses and strains of each UD lamina in the laminate are available, state variables, SDV7, SDV8, SDV15, and SDV17 are calculated, and failures of the fiber, the matrix, the fiber–matrix interface, and the element are checked.

Example problem 3: OHT of \([0/90]_4s, [0/\pm 45/90]_3s\), and \([0/\pm 30/\pm 60/90]_2s\) laminates with \(a/w = 0.15\).

Quasi-static deformations of the three OHT laminates experimentally tested were numerically analyzed using Abaqus/Explicit as for the other two example problems. The FE mesh around the open hole was refined, as shown in Figure 5(b). As mentioned in the subsection “Ultimate failure criteria of a material point,” the mesh size effect is minimized by the using the material length \(l\) and the attenuation coefficient \(d_i\); their values for the 0° and the 90° UD layers are listed.
Figure 4. (a) Sketch of the specimen and the rigid rollers, and their discretization into finite elements. (b) CPU time for different number of elements in each ply and the thickness of the pure resin layer; (c) CPU time for different thickness proportion of the resin layers to that of the UD layers. Through-the-thickness stress response: (d, e, f) $S_{11}$ in $0^\circ$, $S_{22}$ in $90^\circ$ and $S_{12}$ in the resin layers for different number of elements in the thickness direction in each ply and resin layer. (g) Evolution of different energies in the FPB test on $[0^\circ/90^\circ]_{8s}$ laminates.
in Table 6. For different values of \( l \), we have exhibited in Figure 5(c) the final failure load with and without considering \( d_i' \). It is clear that the values of both \( l \) and \( d_i' \) noticeably affect the failure load for the \( 0^\circ \) and the \( 90^\circ \) laminates. For \( d_i' \) unequal to zero, the failure load seems to stabilize for \( l > 0.2 \). We took \( l = 0.319 \) for results presented herein. Values of \( K_T' \) around the open hole in an UD layer are given in Table 7.

We have exhibited time histories of various energies for the OHT \([0^\circ/90^\circ]_4s\) laminates till final failure. It is evident that the balance of energy is well-satisfied lending credence to the computed results.

**Comparison between computed and experimental results**

**Four point bending.** The load versus displacement curves for the six identical specimens and their damage morphology are, respectively, exhibited in Figure 6(a) and (b). It is evident that delamination occurred between adjacent layers at several locations. A few plies cracked near the mid-span. The sequence of ply cracking and delamination could not be determined from the sound recorded during the tests, and no post-mortem examination was conducted to ascertain whether or not failure occurred at interior points of the laminates. The load–displacement curves of Figure 6(a) reveal the failure initiation indicated by a sudden drop in the load with a tiny increase in the axial displacement. A possible reason for the scatter in results of the six replicates is due to the slippage near the lower supporting area. As shown in Figure 6(b), the fracture is concentrated in the fulcrum and the interlayer zone.

As should be clear from plots of the computed and the experimental load versus displacement curves exhibited in Figure 6(a), the present approach gives results close to the test findings. Fringe plots of the von Mises stress at the final load are displayed in Figure 7. In Table 8, we have exhibited fringe plots of the failure indexes for different failure modes at stages I, II, and III corresponding, respectively, to the displacement, \( u_z = -1.0 \text{ mm}, -2.0 \text{ mm}, \) and \(-3.0 \text{ mm}\). These results evince that the damage initiates due to the fiber–matrix interface debonding and the matrix failure in the \( 90^\circ \) layer located at the laminate bottom. The laminate finally fractures when the interlayer delamination occurs at the mid-surface of the laminate. The fiber failure is also seen prior to the ultimate failure. As shown in column II of Table 8, the fiber failure first occurs in area A (grey areas

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**Figure 5.** (a) FE meshes for the simple tension and (b) the OHT laminates.
Figure 6. (a) Load versus displacement curves for six replicates and computed results of the \([0^\circ/90^\circ]_8\) T300/7901 laminates. (b) Photographs of the post-failure unloaded specimens.

Figure 7. Fringe plots of the von Mises stress at the final failure load of \([0^\circ/90^\circ]_8\) laminates.
Table 8. Fringe plots of the progressive damage in FPB of laminates.

<table>
<thead>
<tr>
<th>Fiber–matrix interface debonding in UD layers</th>
<th>Matrix failure in UD layers</th>
<th>Fiber failure in UD layers</th>
</tr>
</thead>
<tbody>
<tr>
<td>I ((u_z = -1.0 , \text{mm}))</td>
<td>II ((u_z = -2.0 , \text{mm}))</td>
<td>III ((u_z = -3.0 , \text{mm}))</td>
</tr>
</tbody>
</table>

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where fiber failure index exceeds 1) of the top and the bottom 0° layers; these zones are close to the four supports. Also, area B between the two lower supports on the bottom 0° layer has high possibility of fiber failure due to the large axial tensile stresses developed there. The same holds on the inner side areas between the two upper rollers of the top 0° layer where large axial compressive stresses have developed.

The matrix failure first occurred in the top and the bottom 90° layers, and the interlayer delamination most likely initiated from the interface layer near the laminate mid-surface. We found a high possibility of delamination in area C located between the supporting and the loading rollers as shown in Figure 2. Prior to delamination occurring, the failure of the 90° layers located on the bottom side had generated voids that accelerated the fracture of the inter-layer surface and led to the final failure of the laminates. We note that stress concentrations occurred in the neighborhood of the loading and the supporting points where material degradation and element deletion were suppressed. Values of different damage parameters in these areas were high signifying failures there. Even though it is difficult to find clues of the damage initiation from the computed nearly linear load–displacement curve and thus hard to compare the experimentally observed damage process with the computed one, the computed load–displacement curve and the final fracture patterns
match well with their experimental counterparts, as shown in Figures 6 and 7.

The computed (experimental average) ultimate displacement of the top roller and the bending force, respectively, equaled 3.1 mm (3.91 mm) and 5469 N (6022 N).

Table 9. Simple tension of laminates.

<table>
<thead>
<tr>
<th>Laminates</th>
<th>Experimental fractures</th>
<th>Computed fractures</th>
<th>Stress–strain curves</th>
</tr>
</thead>
<tbody>
<tr>
<td>[$0^\circ/90^\circ$]$_{4s}$</td>
<td><img src="image1" alt="Image" /></td>
<td><img src="image2" alt="Image" /></td>
<td><img src="image3" alt="Image" /></td>
</tr>
<tr>
<td>[$0^\circ/\pm 45^\circ/90^\circ$]$_{2s}$</td>
<td><img src="image4" alt="Image" /></td>
<td><img src="image5" alt="Image" /></td>
<td><img src="image6" alt="Image" /></td>
</tr>
<tr>
<td>[$0^\circ/\pm 30^\circ/\pm 60^\circ/90^\circ$]$_s$</td>
<td><img src="image7" alt="Image" /></td>
<td><img src="image8" alt="Image" /></td>
<td><img src="image9" alt="Image" /></td>
</tr>
</tbody>
</table>

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Simple tensile deformations of [$0^\circ/90^\circ$]$_{4s}$, [$0^\circ/\pm 45^\circ/90^\circ$]$_{2s}$, and [$0^\circ/\pm 30^\circ/\pm 60^\circ/90^\circ$]$_s$, laminates. Measured and predicted fracture morphology and the axial stress–the axial strain curves of [$0^\circ/90^\circ$]$_{4s}$, [$0^\circ/\pm 45^\circ/90^\circ$]$_{2s}$, and [$0^\circ/\pm 30^\circ/\pm 60^\circ/90^\circ$]$_s$, laminates deformed in ST are shown in Table 9. As seen from column I of Table 9,
Table 10. Fringe plots of the progressive damage in OHT on the \([0^\circ/90^\circ]_{4s}\) laminate.

- **Fiber–matrix interface debonding**
  - I (Axial strain = 0.2%)
  - II (Axial strain = 0.45%)
  - III (Axial strain = 0.675%)

- **Matrix failure**
  - I (Axial strain = 0.2%)
  - II (Axial strain = 0.45%)
  - III (Axial strain = 0.675%)

(continued)
the experimental fracture surface of a laminate varied with the fiber orientation. The \([0^\circ/90^\circ]_{4s}\) laminates showed a straight fracture surface which the computations well duplicated. For the other laminates, the predicted failure surface somewhat differed from that observed in experiments. However, the computed and the experimental stress–strain curves match well with each other for the three laminates, as shown in column III of Table 9. During testing \([0^\circ/\pm45^\circ/90^\circ]_{2s}\) laminates, the recorded cracking sounds are indicated in the figure; however, these could not be related to the damage occurring in the laminate interior. The computed (experimental) ultimate axial stress and axial strain of \([0^\circ/90^\circ]_{4s}\), \([0^\circ/\pm45^\circ/90^\circ]_{2s}\), and \([0^\circ/\pm30^\circ/\pm60^\circ/90^\circ]_{3s}\) laminates, respectively, equaled 953 MPa (1043 MPa) and 1.27% (1.48%), 732 MPa (709 MPa) and 1.45% (1.22%), and 532 MPa (530 MPa) and 1.05% (1.02%).

Simple tensile deformations of the open hole laminates. Computed nominal axial stress versus nominal axial strain (overall elongation divided by the original specimen length) responses of the open hole \([0^\circ/90^\circ]_{4s}\), \([0^\circ/\pm45^\circ/90^\circ]_{2s}\), and \([0^\circ/\pm30^\circ/\pm60^\circ/90^\circ]_{3s}\), laminates deformed due to the axial tensile load applied at the end faces, and the corresponding experimental ones are plotted in Figure 8(a). The stress–strain curves for the three laminates are essentially linear until fracture and the computed ones closely mimic the corresponding experimental ones. The predicted ultimate load is close to that found in tests. The simulations illustrate that the progressive material damage is concentrated near the open hole. The evolution of the material damage is listed in Tables 9 to 12.

The cylindrical coordinates \((r, \theta, z)\) are used to identify failure points. The origin of the cylindrical coordinates is on the laminate mid-plane.
Table 11. Fringe plots of the progressive damage in OHT on the \([0^\circ/\pm 45^\circ/90^\circ]_2s\) laminate.

<table>
<thead>
<tr>
<th>Event</th>
<th>I (Axial strain = 0.2%)</th>
<th>II (Axial strain = 0.463%)</th>
<th>III (Axial strain = 0.725%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber–matrix interface</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>debonding</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Matrix failure</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 11. Continued.

<table>
<thead>
<tr>
<th>Fiber failure</th>
<th>Element failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>I (Axial strain = 0.2%)</td>
<td>I (Axial strain = 0.2%)</td>
</tr>
<tr>
<td>II (Axial strain = 0.463%)</td>
<td>II (Axial strain = 0.463%)</td>
</tr>
<tr>
<td>III (Axial strain = 0.725%)</td>
<td>III (Axial strain = 0.725%)</td>
</tr>
</tbody>
</table>

Table 12. Fringe plots of the progressive damage in OHT on the $[0^\circ/\pm30^\circ/\pm60^\circ/90^\circ]_s$ laminate.
Table 12. Continued.

<table>
<thead>
<tr>
<th>Fiber–matrix interface debonding</th>
<th>Matrix failure</th>
<th>Fiber failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Images showing debonding at different strains]</td>
<td>[Images showing matrix failure at different strains]</td>
<td>[Images showing fiber failure at different strains]</td>
</tr>
<tr>
<td>I (Axial strain = 0.2%)</td>
<td>II (Axial strain = 0.425%)</td>
<td>III (Axial strain = 0.638%)</td>
</tr>
<tr>
<td>[Images showing debonding at different strains]</td>
<td>[Images showing matrix failure at different strains]</td>
<td>[Images showing fiber failure at different strains]</td>
</tr>
<tr>
<td>I (Axial strain = 0.2%)</td>
<td>II (Axial strain = 0.425%)</td>
<td>III (Axial strain = 0.638%)</td>
</tr>
<tr>
<td>[Images showing debonding at different strains]</td>
<td>[Images showing matrix failure at different strains]</td>
<td>[Images showing fiber failure at different strains]</td>
</tr>
<tr>
<td>I (Axial strain = 0.2%)</td>
<td>II (Axial strain = 0.425%)</td>
<td>III (Axial strain = 0.638%)</td>
</tr>
<tr>
<td>(continued)</td>
<td>(continued)</td>
<td>(continued)</td>
</tr>
</tbody>
</table>
Element centroid positions are listed in cylindrical coordinates ($r$, $\theta$, and $z_n$) shown in the first row of Tables 9 to 12 with the origin at the hole center, where $z_n$ denotes the coordinate through the thickness direction of the $n$th UD composite ply from the mid-plane of the laminate and $z_n = 0.125(n-1) + 0.125/2$ mm with $n = 1, 2, \ldots, 8$.

We see from results listed in Tables 9 to 12 that for each one of the three laminates, fiber breakage occurred in the $0^\circ/C_{14}$ layer. It initiated from the edge of the open hole symmetrically along the horizontal direction that is normal to the loading direction and where the maximum axial tensile stress occurred due to the stress concentration effect. The matrix first failed in a $90^\circ/C_{14}$ layer in Figure 8. (a) Axial stress–axial strain curves for OHT of $[0^\circ/90^\circ]_4s$, $[0^\circ/\pm45^\circ/90^\circ]_2s$, and $[0^\circ/\pm30^\circ/\pm60^\circ/90^\circ]_b$ laminates with $a/w = 0.15$. (b) Axial stress–axial strain curves for layers with different fiber orientation from results computed for the OHT $[0^\circ/\pm45^\circ/90^\circ]_2s$ laminates with $a/w = 0.15$.

<table>
<thead>
<tr>
<th>Element failure</th>
<th>(Axial strain = 0.2%)</th>
<th>(Axial strain = 0.425%)</th>
<th>(Axial strain = 0.638%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 12. Continued.</td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Axial strain (%)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.425%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.638%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Figure 9. (a) Fracture morphology of (i) \( [0/90]_{4s} \), (ii) \( [0/\pm45/90]_{2s} \), and (iii) \( [0/\pm30/\pm60/90] \), laminates. (b) Element failure propagation in the \( [0/90]_{4s} \) laminate: (i) axial strain = 0.46%, (ii) axial strain = 0.70%, and (iii) axial strain = 0.80%. (c) Element failure propagation in the \( [0/\pm45/90]_{2s} \) laminate: (i) axial strain = 0.47%, (ii) axial strain = 0.72%, and (iii) axial strain = 0.80%. (d) Element failure propagation in the \( [0/\pm30/\pm60/90] \) laminate: (i) axial strain = 0.43%, (ii) axial strain = 0.64%, and (iii) axial strain = 0.70%.
Table 13. Summary of observation derived from the computed results.

<table>
<thead>
<tr>
<th>Failure Mode</th>
<th>[0°/90°]4s</th>
<th>[0°/±45°/90°]2s</th>
<th>[0°/±30°/±60°/90°]4s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber–matrix debonding</td>
<td>First occurs in elements (3 mm, 74.3°, z₀), (3 mm, 105.7°, z₁), (3 mm, 254.3°, z₂) and (3 mm, 285.7°, z₃) in the 90° plies, n = 1, 3, 5, 7.</td>
<td>First occurs in elements (3 mm, 95.2°, z₀), (3 mm, 102.2°, z₁), (3 mm, 275.2°, z₂) and (3 mm, 282.2°, z₃) in the 90° plies.</td>
<td>First occurs in elements (3 mm, 94.1°, z₁) and (3 mm, 274.1°, z₁) in the 90° plies.</td>
</tr>
<tr>
<td>Matrix failure</td>
<td>First occurs in elements (3 mm, 90°, z₀) and (3 mm, 270°, z₁) in the 90° plies, n = 1, 3, 5, 7.</td>
<td>First occurs in elements (3 mm, 91.8°, z₀), (3 mm, 98.7°, z₁), (3 mm, 271.8°, z₂) and (3 mm, 278.7°, z₃) in the 90° plies.</td>
<td>First occur in elements (3 mm, 94.1°, z₁) and (3 mm, 274.1°, z₁) in the 90° plies.</td>
</tr>
<tr>
<td>Fiber failure</td>
<td>First occurs in elements (3 mm, 90°, z₀) and (3 mm, 270°, z₁) in the 90° plies, n = 2, 4, 6, 8.</td>
<td>First occurs in elements (3 mm, 90°, z₀) and (3 mm, 270°, z₁) in 90° plies, n = 4, 8.</td>
<td>First occurs in elements (3 mm, 90°, z₀) and (3 mm, 270°, z₁) in 90° plies.</td>
</tr>
<tr>
<td>Element failure</td>
<td>First occurs in elements (3 mm, 90°, z₀) and (3 mm, 270°, z₁) in the 90° plies, n = 1, 3, 5, 7.</td>
<td>First occurs in elements (3 mm, 91.8°, z₀), (3 mm, 98.7°, z₁), (3 mm, 271.8°, z₂) and (3 mm, 278.7°, z₃) in the 90° plies.</td>
<td>First occurs in elements (3 mm, 94.1°, z₁) and (3 mm, 274.1°, z₁) in the 90° plies.</td>
</tr>
<tr>
<td>Ultimate load, N</td>
<td>8632</td>
<td>6716</td>
<td>6105</td>
</tr>
<tr>
<td>Ultimate axial strain, % (computed/experimental)</td>
<td>0.71/0.72</td>
<td>0.72/0.77</td>
<td>0.63/0.83</td>
</tr>
<tr>
<td>Ultimate strength, MPa (computed/experimental)</td>
<td>432/505</td>
<td>336/364</td>
<td>305/334</td>
</tr>
<tr>
<td>Number of failed elements at the peak load</td>
<td>385</td>
<td>249</td>
<td>195</td>
</tr>
<tr>
<td>Work done by the external force, mj</td>
<td>1996</td>
<td>1574</td>
<td>1236</td>
</tr>
<tr>
<td>Total strain energy, mj</td>
<td>1990</td>
<td>1570</td>
<td>1234</td>
</tr>
</tbody>
</table>

the three laminates and was followed by that in the ±60°, ±45°, ±30°, and 0° layers sequentially.

In Tables 9 to 12, we have exhibited fringe plots of different damage modes as viewed from the top of the 0° layer. It should be noted that the UD layers with the same fiber orientation but in different laminates could be differently damaged. In order to gain a better insight into the damage mode of each layer, fringe plots in a small area around the hole are exhibited in Tables 9 to 12. The grey areas with a failure index greater than 1 indicate that the material there has been damaged. The fiber–matrix interface debonding occurred at an early stage of the loading that accelerated the matrix failure. Elements in the 90° layers first satisfied the ultimate failure criterion and were deleted, as shown in Figure 9. Subsequently, elements in the UD layer with smaller fiber off-axis angle were deleted. The deletion of these elements did not affect much the slope of the axial stress versus the axial strain response, since fibers along the loading axis mostly supported the applied tensile traction. However, once an element in these layers completely failed and was deleted, the entire laminate quickly failed. In Table 13, we have summarized details of the damage initiation location of each failure mode.

Conclusions

We have experimentally studied the failure of UD fiber-reinforced T300/7901 laminates under four point bending (FPB), simple tensile (ST) loading, and simple tensile loading of laminates with open holes at their centroids (OHT). We have developed a mathematical framework for analyzing the intra-layer and the inter-layer damage of these laminates and implemented it in the commercial FE software, Abaqus. The problem formulation assumes that a fiber is transversely isotropic and linearly elastic. The matrix is isotropic and deforms elasto-plastically obeying the Drucker–Prager yield criterion. Failure modes considered are fiber breakage, fiber/matrix debonding, delamination, and matrix failure. Huang’s bridging model was used to determine
constituent-level stresses from the macroscopic stresses to employ constituent level failure criteria. The delamination is simulated by considering the failure of a thin pure resin layer inserted between two adjacent plies. For all laminates studied, both the computed and the measured axial stress versus the axial strain responses under FPB, ST, and ST of laminates with open holes at their centroids were essentially linear until their ultimate failure and were close to each other. This was because all had some 0° layers along the loading direction that were the last to be damaged before the ultimate failure.

Declaration of Conflicting Interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The research received financial support and computational facilities provided by the Department of Biomedical Engineering and Mechanics, Virginia Polytechnic Institute and State University (VT) while Yuqing Zhao was visiting RCB’s research group, and financial supports from the National Natural Science Foundations of China with grant numbers 11832014 and 11472192. RCB’s work was partially supported by the US Office of Naval Research grant N00014-1-18-2538 to VT.

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RC Batra https://orcid.org/0000-0002-7191-2547

References


61. Gopinath G and Batra RC. Sensitivity of responses of three micro-mechanics approaches to changes in unit

Appendix 1: Stress concentration factors

\[ K_{22}(\phi) = \left\{ 1 + \frac{A}{2} \sqrt{V_f \cos 2\phi + \frac{B}{2(1 - \sqrt{V_f})}} \left[ V_f^2 \cos 4\phi + 4V_f(\cos 2\phi)^2(1 - 2\cos 2\phi) + \sqrt{V_f}(2\cos 2\phi + 4\cos 4\phi) \right] \right\} \times \left( V_f + 0.3V_m \right) E_{12}^f + 0.7V_mE_m \] 
\[ = \frac{2E_m^fE_m^f(\frac{\sqrt{\nu}}{2})^2 + E_{11}^fE_m^f(\nu_{21} - 1) - E_{22}^f[2(\nu_{12}^m)^2 + \nu_{12}^m - 1]}{E_{11}^fE_{22}^f + E_m^f(1 - \nu_{21}) + E_mE_m - 2E_{22}^fE_m(\nu_{12}^m)^2} \] 

\[ A = \frac{2E_m^fE_m^f(\frac{\sqrt{\nu}}{2})^2 + E_{11}^fE_m^f(\nu_{21} - 1) - E_{22}^f[2(\nu_{12}^m)^2 + \nu_{12}^m - 1]}{E_{11}^fE_{22}^f + E_m^f(1 - \nu_{21}) + E_mE_m - 2E_{22}^fE_m(\nu_{12}^m)^2} \] 

\[ B = \frac{E_m^f(\nu_{21} + 1) - E_{22}^f[\nu_{12}^m + 1]}{E_{22}^f[\nu_{12}^m + 4(\nu_{12}^m)^2 - 3] - E_m^f(\nu_{23} + 1)} \]
\[
K_{t2} = K_{22}, \quad K_{c2} = K_{c2} \quad (18, 19)
\]

For details of \( \tilde{K}_{t2} \), see Ramkumar et al.\(^{76}\)

**Appendix 2**

The explicit expressions of the moduli of the UD composite, derived from constituents’ properties are

\[
E_{11} = V_f E_{f1} + V_m E_m,
\]

\[
E_{22} = \frac{(V_f + V_m a_{11})(V_f + V_m a_{22})}{(V_f + V_m a_{11})(V_f + V_m a_{22}) + V_f V_m (S_{22}^m - S_{21}^m) a_{12}}
\]

\[
K_{33} = K_{22}^{c}, \quad K_{23}^{c} = 2\sigma_{m,2}^m \sqrt{K_{22}^c K_{22}^c \frac{K_{22}^c}{\sigma_{m,2}^m \sigma_{m,2}^m}}
\]

\[
W(V_f) = \pi \sqrt{V_f} \left[ \frac{1}{4V_f} - \frac{1}{32} \frac{1}{256} V_f - \frac{5}{4096} V_f^2 \right]
\]

\[
K_{12} = \left[ \frac{1}{1 - V_f G_{12}^f - G_m^m} \left( W(V_f) - \frac{1}{3} \right) \right]
\]

\[
K_{12} = \frac{(V_f + 0.3 V_m) G_{12}^f + 0.7 V_m G_m^m}{0.3 G_{12}^f + 0.7 G_m^m}
\]
\[ \nu_{12} = \frac{V_f + V_m a_6}{V_f/G_{12} + V_m a_6/G_m}, \]

\[ G_{23} = \frac{V_f + V_m a_{22}}{V_f/G_{23} + V_m a_{22}/G_m} \]  

According to equation 7(a) to (f), the degradation of \( E_{11} \) and \( E_{22} \) due to the damage induced in the matrix and the fiber failure under uniaxial tension are exhibited in Figures 10 and 11. It is evident that the matrix failure has a small effect on \( E_{11} \) because of very small value of the matrix modulus as compared to that of the fiber. However, the transverse modulus, \( E_{22} \), is decreased by a factor of 4 due to the matrix failure. The value of \( m \) noticeably affects the decay rate of these moduli with \( m = 0.2 \) having a rapid decrease and \( m > 1 \) giving a gradual decrease. As expected, the fiber failure sharply decreases both \( E_{11} \) and \( E_{22} \).