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STRESS DISTRIBUTION IN AN ELASTIC PERFECTLY PLASTIC PLATE SUBJECTED TO CORROSIVE ENVIRONMENTAL LOADS

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Abstract—We analyse deformations of an isotropic elastic/perfectly plastic plate subjected to environmental effects such as the corrosive forces exerted by the surrounding medium. It is found that for the bounding surfaces of the plate to deform plastically, the corrosion process must propagate to a point whose distance from the outer bounding surface exceeds one third the half-thickness of the plate, and for the central unaffected material to also deform plastically the half-thickness of the corroded layer must exceed five eighths the half-thickness of the plate.

INTRODUCTION

The mechanical behavior of materials subjected to reactive environments is of interest in many industries such as the electronics industry wherein dies are etched by dipping the body in a reactive chemical solution. Often, one deals with small scale structures, and the surrounding medium can cause severe adverse effects leading to its eventual failure. If the forces exerted by the environment on the structure could be estimated with some certainty, then the problem of analyzing deformations of the structure will reduce to solving an initial-boundary-value problem. However, such information is generally lacking. Therefore, we use here a semi-inverse approach in the sense that we represent deformations caused by the corrosive medium by an eigenstrain, motivate a reasonable expression for it, and ascertain conditions under which the body will deform elastically and/or plastically.

Here we consider a flat plate made of an isotropic elastic/perfectly plastic material. We envisage that the plate is initially stress free and is exposed to a reactive environment. The corrosion process affects the faces of the plate by, for example, selective removal of atoms so that pores are generated within the plate, and the pore concentration varies through the thickness, being highest at the outermost surface layer in contact with the environment and gradually decreasing to zero. An example of such a process is the preferential dissolution of copper from Cu_3Al alloy in NaCl solution [1]. The corrosion affected layer tends to contract. Here we analyze the stress distribution within the plate because of the differential contraction of various layers.

FORMULATION OF THE PROBLEM

In order to simplify the problem, we consider an infinite plate of thickness $2h$ and made of an isotropic elastic/perfectly plastic material. We use rectangular Cartesian coordinates with origin at the mid surface of the plate and x_3-axis perpendicular to its faces. The bounding surfaces $x_3 = \pm h$ of the plate are exposed to a corrosive environment and are presumed to be traction free. The plate layer in contact with the environment deforms because of the exchange of particles between the plate and the environment. These deformations induce stresses in the plate.

We assume that the plate deforms quasistatically so that inertia effects are negligible, a state of plane stress prevails within the plate, corrosive forces cause only normal stresses, the deformations of the plate are infinitesimal in the sense that linear kinematical relations can be used, and all deformation variables are functions of x_3 and time t. Thus,

$$\sigma_{ij} = \sigma(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) \tag{1}$$

where δ_{ij} is the Kronecker delta. If the plate were anisotropic, the corrosive forces will likely induce shear stresses too, and equation (1) will need to be appropriately modified. The stress state (1) satisfies identically the equilibrium equations

$$\sigma_{ij,j} = 0, \tag{2}$$

where a comma followed by index j indicates partial differentiation with respect to x_j, and a repeated index implies summation over the range of the index. The pertinent boundary conditions are

$$\sigma_{i3}(x_3 = \pm h, t) = 0, \qquad \int_{-h}^{h} \sigma \, dx_3 = 0, \qquad \int_{-h}^{h} x_3 \sigma \, dx_3 = 0. \tag{3}$$

That is, the top and bottom surfaces of the plate are traction free, and the resultant forces and moments on any section of the plate vanish. The presumed stress state (1) satisfies the boundary condition $(3)_1$ for all values of σ.

We assume that the material obeys von Mises yield criterion which, for the stress state (1), reduces to

$$\sigma^2 = Y^2 \tag{4}$$

where Y is the yield stress in a quasistatic simple tension or compression test.

Our goal is to find a nontrivial solution of equations (1) and (3) which accounts for the deformations caused by the corrosive environment, and identify regions of the plate material that at a given instant of time are deforming elastically or plastically.

SOLUTION OF THE PROBLEM

The stress state (1) and the stress–strain relations for isotropic elastic/perfectly plastic materials imply that the only nonzero components of the infinitesimal strain tensor ϵ_{ij} are ϵ_{33} and

$$\epsilon_{11} = \epsilon_{22} = \epsilon. \tag{5}$$

The strain field will satisfy the compatibility conditions if and only if

$$\epsilon_{,33} = \frac{\partial^2 \epsilon}{\partial x_3^2} = 0. \tag{6}$$

Thus,

$$\epsilon(x_3, t) = F_1(t) + x_3 F_2(t) \tag{7}$$

where F_1 and F_2 are functions of time t, satisfies the compatibility condition (6). The functions F_1 and F_2 are to be determined so that boundary conditions (3) are satisfied.

In order to delineate whether a material point is deforming elastically or plastically, we work below in terms of strain-rate rather than strain, and assume that at any time t, it has the decomposition [2]

$$\dot{\epsilon} = \dot{\epsilon}^e + (1 - g)\dot{\epsilon}^p + \dot{\epsilon}^* \tag{8}$$

where ϵ^e, ϵ^p, and ϵ^* are elastic, plastic, and eigen strains, respectively, and a superimposed dot indicates the time derivative. The eigenstrain tensor ϵ^* gives the local constraint-free deformation of the corroded or affected layer. However, the actual deformation is not ϵ^* due to the substrate constraining [3]. The function $g(x_3, t)$ is defined as

$$g(x_3, t) = 0 \quad \text{if the material point is deforming plastically at time } t, \text{ and}$$
$$g(x_3, t) = 1 \quad \text{otherwise.} \tag{9}$$

For a point (x_3, t) deforming elastically, we have

$$\dot{\sigma}^e(x_3, t) = \frac{E}{(1-\nu)}[\dot{F}_1 + x_3\dot{F}_2 - \dot{\epsilon}^*] \tag{10}$$

where E is Young's modulus, and ν is Poisson's ratio for the plate material, and we have used equation (7) and Hooke's law. However, if a material point is deforming plastically, then the yield condition (4) is satisfied, and we have

$$\dot{\sigma} = 0 \quad \text{when} \quad g(x_3, t) = 0. \tag{11}$$

Since a material point is deforming either elastically or plastically, we can combine equations (10) and (11) into the following

$$\dot{\sigma}(x_3, t) = \frac{Eg}{1-\nu}[\dot{F}_1 + x_3\dot{F}_2 - \dot{\epsilon}^*]. \tag{12}$$

Recalling that the plate is initially stress free, equations $(3)_2$ and $(3)_3$ are equivalent to

$$\int_{-h}^{h} \dot{\sigma}(x_3, t)\,\mathrm{d}x_3 = 0, \qquad \int_{-h}^{h} x_3\dot{\sigma}(x_3, t)\,\mathrm{d}x_3 = 0, \tag{13}$$

which, when combined with equation (12), give

$$h_0\dot{F}_1 + h_1\dot{F}_2 = \int_{-h}^{h} g\dot{\epsilon}^*\,\mathrm{d}x_3,$$
$$h_1\dot{F}_1 + h_2\dot{F}_2 = \int_{-h}^{h} x_3 g\dot{\epsilon}^*\,\mathrm{d}x_3, \tag{14}$$

where

$$h_\alpha = \int_{-h}^{h} x_3^\alpha g\,\mathrm{d}x_3, \qquad \alpha = 0, 1, 2. \tag{15}$$

If $\dot{\epsilon}^*$ were known, then equations (12) and (14) could be solved for g and $\dot{\sigma}$ and, hence, σ from

$$\sigma = \int_0^t g\dot{\sigma}\,\mathrm{d}t, \tag{16}$$

with

$$g(x_3, t) = 0 \quad \text{for} \quad \sigma^2 = Y^2, \quad \sigma\dot{\sigma} > 0,$$
$$g(x_3, t) = 1 \quad \text{for} \quad \sigma^2 < Y^2 \quad \text{or} \quad \sigma^2 = Y^2 \quad \text{and} \quad \sigma\dot{\sigma} \le 0. \tag{17}$$

For a general $\dot{\epsilon}^*$, equations (12) and (14) need to be integrated with respect to time t by using a numerical method such as the forward difference method. However, if the eigenstrain function ϵ^* is simple enough so that it is possible to predict *a priori* the plastic zone, then these equations can be integrated analytically.

Assuming that no unloading occurs in the time interval $0 \le t \le T$, and recalling that the plate is initially stress free, then

$$\sigma(x_3, t) = \sigma^e(x_3, t) \quad \text{for} \quad g(x_3, t) = 1,$$
$$\sigma(x_3, t) = (\operatorname{sgn}\sigma)Y \quad \text{for} \quad g(x_3, t) = 0, \tag{18}$$

where

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{for} \quad \sigma > 0, \\ -1 & \text{for} \quad \sigma < 0, \end{cases} \tag{19}$$

and boundary conditions $(3)_2$ and $(3)_3$ become

$$\int_{-h}^{h} [g\sigma + (1-g)(\operatorname{sgn}\sigma)Y]\,dx_3 = 0, \tag{20}$$

$$\int_{-h}^{h} x_3[g\sigma + (1-g)(\operatorname{sgn}\sigma)Y]\,dx_3 = 0, \tag{21}$$

with

$$\sigma^e = \frac{E}{(1-\nu)}[F_1 + F_2 - \epsilon^*]. \tag{22}$$

Henceforth, we work with equations (18)–(22).

We now choose the eigenstrain function ϵ^* to model a corrosion process. It is reasonable to assume that the corrosion process proceeds symmetrically from the outer bounding surfaces into the plate, and at time t has progressed to the point for which $x_3 = \pm l$. The eigenstrain should be maximum at the outer bounding surfaces and should decrease gradually to zero at $x_3 = \pm l$. A reasonable expression for ϵ^* satisfying these conditions is

$$\epsilon^*(x_3, t) = \begin{cases} -\gamma\left(\dfrac{h - l(t)}{h}\right)\left(\dfrac{x_3 + l(t)}{h}\right)^2, & -h \le x_3 \le -l, \\ 0 & |x_3| < l, \\ -\gamma\left(\dfrac{h - l(t)}{h}\right)\left(\dfrac{x_3 - l(t)}{h}\right)^2, & l \le x_3 \le h, \end{cases} \tag{23}$$

where γ is a positive constant to be determined from the test data. Since the plate is being deformed by corrosive forces only and ϵ^* is taken to be symmetric in x_3, it follows that $g(x_3, t)$ is also symmetric in x_3. Thus, equations (18)–(22) give

$$\begin{aligned} F_1 &= \frac{2}{h_0}\int_0^h g\epsilon^*\,dx_3 - \frac{2}{h_0}\frac{1-\nu}{E}\int_0^h (1-g)(\operatorname{sgn}\sigma)Y\,dx_3, \\ F_2 &= 0, \\ \sigma &= \frac{E}{1-\nu}\left[\frac{2}{h_0}\int_0^h g\epsilon^*\,dx_3 - \epsilon^*\right] - \frac{2}{h_0}\int_0^h (1-g)(\operatorname{sgn}\sigma)Y\,dx_3. \end{aligned} \tag{24}$$

When the corrosive environment removes material from the plate, the eigenstrain near the bounding surfaces of the plate will be compressive and the resulting stresses will be tensile. However, the stresses within the central portion will be tensile in order to make the resultant force across the plate equal to zero. If there is any plasticity induced, it will first occur at the outer faces and then, as the affected layer grows, the central part of the plate may also be deformed plastically. We investigate below conditions for the plate to be deformed plastically.

First, consider the case when the entire plate thickness is deformed elastically so that $g(x_3, t) = 1$ for $-h \le x_3 \le h$ and $0 \le t \le T$. Then, for a fixed l and time t, the stress distribution within the plate is given by

$$\bar{\sigma} = \begin{cases} -\dfrac{\bar{\gamma}}{3}(1-\bar{l})^4, & 0 \le \bar{x}_3 \le \bar{l}, \\ \bar{\gamma}(1-\bar{l})\left[(\bar{x}_3 - \bar{l})^2 - \dfrac{1}{3}(1-\bar{l})^3\right], & \bar{l} \le \bar{x}_3 \le 1, \end{cases} \tag{25}$$

where

$$\bar{\sigma} = \frac{\sigma}{Y}, \quad \bar{\gamma} = \frac{E\gamma}{Y(1-\nu)}, \quad \bar{x}_3 = \frac{x_3}{h}, \quad \bar{l} = \frac{l}{h}$$

are nondimensional quantities.

Henceforth, we use nondimensional variables only and drop the superimposed bars.

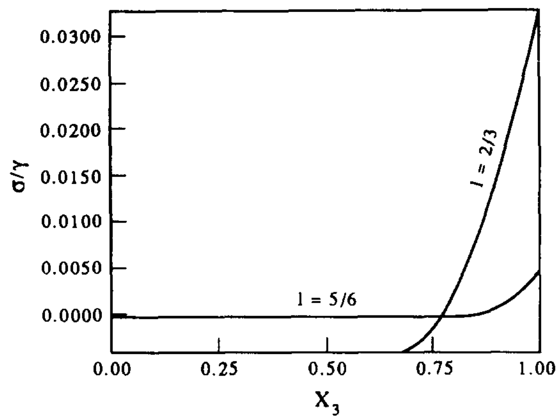

Fig. 1. Stress distribution within half the plate thickness when the corrosion process has progressed from the outer bounding surface to 1/6 and 1/3 the half-thickness of the plate.

If the thickness of the corroded layer is small, say $1-l=1/6$, then the tensile stress generated at the bounding surfaces is approximately 16 times that at the plate center. As the corrosion penetrates into the plate, larger compressive stresses are induced within the central portion, and the difference between the magnitudes of the tensile stress at the bounding surfaces and the compressive stress within the central portion of the plate decreases. Figure 1 depicts the stress distribution through the thickness of the plate for $(1-l)=1/6$ and 1/3.

For plastic yielding to commence at the bounding surfaces of the plate, $\sigma=1$ at $x_3=1$ in equation (25), and we get

$$\gamma=\frac{3}{(1-l)^3(2+l)}. \tag{26}$$

The right-hand side of equation (26) is the minimum value of γ for the plastic yielding to begin at the bounding surfaces of the plate. When plastic yielding has progressed to a point for which $x_3=p$, that is, through a thickness equal to $(1-p)$ from the outer bounding surfaces, the stress distribution is given by

$$\sigma=\begin{cases}-\left[\dfrac{\gamma(1-l)}{3p}(p-l)^3+\dfrac{1-p}{p}\right], & 0\leq x_3\leq l,\\ \gamma(1-l)\left[(x_3-l)^2-\dfrac{1}{3p}(p-l)^3\right]+1-\dfrac{1}{p}, & l<x_3<p,\\ 1, & p\leq x_3\leq 1.\end{cases} \tag{27}$$

The stress distribution through the thickness of the plate is depicted in Fig. 2. In order to determine p, we set $x_3=p$ and $\sigma=1$ in equation (27) with the following result

$$(p-l)^2(2p+l)=\frac{3}{\gamma(1-l)} \tag{28}$$

which has only one real solution for

$$\gamma<\frac{3}{l^3(1-l)}. \tag{29}$$

From equations (26) and (29), we obtain that the thickness of the corroded layer must exceed 1/3 in order for the bounding surfaces of the plate to begin deforming plastically. The size of the plastic zone, which is determined from equation (28), satisfies the inequality

$$\frac{3}{2}l<p<1. \tag{30}$$

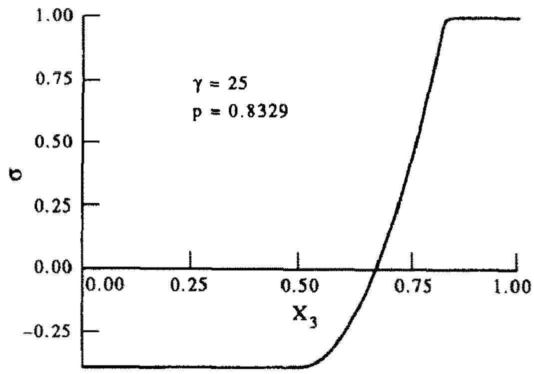

Fig. 2. Stress distribution within half the plate thickness when corrosion has progressed deep enough to cause plastic deformations of the outer layer.

The maximum compressive stress occurs at $x_3=0$. For plastic yielding to occur at the plate center, we set $x_3=0$ and $\sigma=-1$ in equation $(27)_1$. The resulting equation, together with equation (28), gives the following system of equations for p and γ

$$(p-l)^2(2p+l)=\frac{3}{\gamma(1-l)},$$

$$(p-l)^3=\frac{3}{\gamma(1-l)}(2p-1) \tag{31}$$

which have a physically meaningful solution for p and l only if

$$\gamma \geq \frac{32}{9(1-l)(1-2l)^2}. \tag{32}$$

Also,

$$p=\frac{3-2l}{4}>\frac{3}{2}l \tag{33}$$

gives $l<3/8$ for plastic flow to occur at the plate center. The stress distribution in this case is

$$\sigma=\begin{cases} -1, & 0\leq x_3\leq l, \\ \gamma(1-l)\left[(x_3-l)^2-\dfrac{1}{3p}(p-l)^3\right]+1-\dfrac{1}{p}, & l<x_3<p, \\ 1, & p\leq x_3\leq 1. \end{cases} \tag{34}$$

DISCUSSION AND CONCLUSIONS

We have analyzed stress distribution in a plate subjected to corrosive environmental loads which change its material structure by diffusion of atomic particles out of the body and, therefore, induce a strain in it. We have assumed that the plate is made of an isotropic elastic/perfectly plastic material, and that it deforms quasistatically so that inertia effects are negligible. It has been found that when the total thickness of the affected layer is less than one third of the plate thickness, the plate deforms elastically and the magnitude of the tensile stress generated at the bounding surfaces of the plate exceeds sixteen times the compressive stress induced in the unaffected central portion of the plate. This stress distribution supports the Sieradzki and Newman [4, 5] model for the embrittlement of nanoscale structures.

As the affected layer grows, the difference between the magnitudes of the maximum tensile stress in the affected layer and the peak compressive stress in the central unaffected portion of the plate decreases. Once the total thickness of the affected layers exceeds one-third the plate thickness, the bounding surfaces of the plate start deforming plastically, and the plastic flow propagates inward. The half-thickness of this plastically deformed layer is given by

$$\frac{3}{2}l < p < h$$

where l is the half-thickness of the affected layer and $2h$ equals the plate thickness. The central portion of the plate starts deforming plastically in compression when the corrosion process has progressed to a point whose distance from the outer bounding surface exceeds $(5/8)h$.

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