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Dynamic fracture of a Kane–Mindlin plate

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Abstract

We study dynamic crack problems for an elastic plate by using Kane–Mindlin's kinematic assumptions. The general solutions of the Laplace transformed displacements and stresses are first derived. Path independent integrals for stationary cracks subjected to transient loads and steadily growing cracks are deduced. For a stationary crack in a very thin plate subjected to impact loads, the crack tip dynamic stress intensity factor (DSIF), $K_{\mathrm{I}}(t)$, is related to the far field plane stress one, $K_{\mathrm{I}}^{0}(t)$, by $K_{\mathrm{I}}(t)=K_{\mathrm{I}}^{0}(t)/\sqrt{1-\nu^2}$ where ν is Poisson's ratio. For a crack steadily growing with speed V, the crack tip DSIF, $K_{\mathrm{I}}(V)$, is given by $K_{\mathrm{I}}(V)=\sqrt{B(V)/(A(V)(1-\nu^2))}\,K_{\mathrm{I}}^{0}(V)$ where $K_{\mathrm{I}}^{0}(V)$ is the plane stress DSIF and $A(V)$ and $B(V)$ are known functions of V. These results are applied to compute the DSIF for a semi-infinite stationary crack in an unbounded plate subjected to impact pressure on the crack faces. The results of DSIF for a finite crack in an infinite plate under uniform impact pressure on the crack surfaces show that for each plate thickness, the maximum DSIF is higher than that for the plane stress case.

1. Introduction

It is well known that the plane stress assumptions i.e. the transverse stresses are negligible as compared with the in-plane ones, can be used to study deformations of a thin plate subjected to in-plane loads. The governing equations for the two in-plane displacements are then deduced; the transverse displacement can be obtained from the in-plane stresses by using the constitutive relations. Generally speaking, the plane stress theory yields acceptable results for a plate with the thickness at least an order of magnitude smaller than a characteristic inplane dimension. When a plate with a notch is considered, however, the plane stress theory frequently gives incorrect results. As an example, the plane stress results for an elliptical notch problem will be incorrect if the plate thickness is of the order of the minor diameter of the notch. Furthermore, the plane stress theory always fails at the crack tip region as the crack tip can be viewed as a limiting case of an elliptical notch with zero minor diameter. When a plate is subjected to dynamic loads, another length parameter, the wave length, must be

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considered. The plane stress theory will also become invalid if the wave lengths are of the order of the plate thickness. In fact, the deformations in the crack tip region are three-dimensional. However, a full three-dimensional analysis is difficult, particularly for dynamic crack problems. Proposed in [1] is a quasi-three-dimensional theory to study extensional vibrations of a plate. In this theory, the transverse displacement is independent of the in-plane deformations and the final governing equations are of order six instead of four as in plane elasticity. This requires three conditions on the boundary which are consistent with three dimensional elasticity theory. By using the Kane–Mindlin theory, the stress intensity factor for a crack subjected to incident harmonic waves was calculated [2]. The Kane–Mindlin theory has also been used to study static crack problems [3] and interface crack problems [4,5].

We use the Kane–Mindlin theory to study dynamic fracture of a thin elastic plate. The general solutions of the Laplace transformed displacements and stresses are first derived. Path-independent integrals for both stationary cracks subjected to transient loads and steadily growing cracks are deduced. Deformations around a stationary crack subjected to impact pressure on the crack faces are investigated and the effect of the plate thickness on the dynamic stress intensity factor is studied.

2. Kane–Mindlin plate equations and solutions in the Laplace transform plane

Consider a plate of thickness $2h$ and denote by x_1, x_2, x_3 the rectangular Cartesian coordinate system with $x_3 = \pm h$ describing the bounding plate surfaces. The plate is subjected to symmetrical loads about the plane $x_3 = 0$ (the antisymmetrical loads will cause bending which is not considered herein). The theory in Ref. [1] makes the following assumptions on the displacement fields in the plate

$$u_1(x_1, x_2, x_3, t) = v_1(x_1, x_2, t), \qquad u_2(x_1, x_2, x_3, t) = v_2(x_1, x_2, t),$$
$$u_3(x_1, x_2, x_3, t) = (x_3/h) v_3(x_1, x_2, t). \tag{1}$$

Here, u_1, u_2 and u_3 denote displacements of a point in the x_1-, x_2- and x_3-directions, respectively, and $v_3(x_1, x_2, t)$ is the transverse displacement of a point on the surface $x_3 = h$.

Introduce the following stress and strain resultants

$$\{N_{\alpha\beta}(x_1, x_2, t), N_{33}(x_1, x_2, t)\} = \frac{1}{2h}\int_{-h}^{h}\{\sigma_{\alpha\beta}(x_1, x_2, x_3, t), \sigma_{33}(x_1, x_2, x_3, t)\}\,\mathrm{d}x_3,$$
$$\{\gamma_{\alpha\beta}(x_1, x_2, t), \gamma_{33}(x_1, x_2, t)\} = \frac{1}{2h}\int_{-h}^{h}\{\varepsilon_{\alpha\beta}(x_1, x_2, x_3, t), \varepsilon_{33}(x_1, x_2, x_3, t)\}\,\mathrm{d}x_3,$$
$$\{R_\alpha(x_1, x_2, t), \Gamma_\alpha(x_1, x_2, t)\} = \frac{1}{2h}\int_{-h}^{h} x_3\{\sigma_{\alpha 3}(x_1, x_2, x_3, t), 2\varepsilon_{\alpha 3}(x_1, x_2, x_3, t)\}\,\mathrm{d}x_3 \tag{2}$$

where σ_{ij} and ϵ_{ij} are the components of the stress and infinitesimal strain tensors, respectively, indices i and j take values 1, 2, 3 and Greek indices α and β take values 1 and 2. The basic equations for the displacements $\boldsymbol{v}$ are

$$\mu\nabla^2 v_1 + (\lambda+\mu)e_{,1} + \frac{\kappa\lambda}{h}v_{3,1} = \rho\frac{\partial^2 v_1}{\partial t^2}, \qquad \mu\nabla^2 v_2 + (\lambda+\mu)e_{,2} + \frac{\kappa\lambda}{h}v_{3,2} = \rho\frac{\partial^2 v_2}{\partial t^2},$$
$$\mu\nabla^2 v_3 - \frac{3\kappa\lambda}{h}e - \frac{3\kappa^2}{h^2}(\lambda+2\mu)v_3 = \rho\frac{\partial^2 v_3}{\partial t^2} \tag{3}$$

where

$$e = v_{\alpha,\alpha}, \quad \nabla^2(\cdot) = (\cdot)_{,\alpha\alpha}, \tag{4}$$

λ and μ are Lamé constants, ρ is the mass density and a comma followed by the index α implies partial differentiation with respect to x_α, repeated indices imply summation and $\kappa = \pi/2\sqrt{3}$ describes better the strain energy at both low and high frequency vibrations of the plate [1].

The average strains in Eq. (2) are related to the displacements $\boldsymbol{v}$ by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \qquad \gamma_{33} = \frac{v_3}{h}, \qquad \Gamma_\alpha = \frac{h}{3} v_{3,\alpha} \tag{5}$$

and the stress resultants are given by

$$N_{\alpha\beta} = 2\mu\gamma_{\alpha\beta} + \lambda(e + \kappa\gamma_{33})\delta_{\alpha\beta}, \qquad N_{33} = 2\mu\kappa^2\gamma_{33} + \lambda\kappa(e + \kappa\gamma_{33}), \qquad R_\alpha = \mu\Gamma_\alpha \tag{6}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

By taking the Laplace transform of Eq. (3) and assuming zero initial conditions at $t = 0$, we have

$$\mu\nabla^2 v_1^* + (\lambda + \mu)e_{,1}^* + \frac{\kappa\lambda}{h} v_{3,1}^* = \rho p^2 v_1^*, \qquad \mu\nabla^2 v_2^* + (\lambda + \mu)e_{,2}^* + \frac{\kappa\lambda}{h} v_{3,2}^* = \rho p^2 v_2^*,$$

$$\mu\nabla^2 v_3^* - \frac{3\kappa\lambda}{h} e^* - \frac{3\kappa^2}{h^2}(\lambda + 2\mu)v_3^* = \rho p^2 v_3^* \tag{7}$$

where $\boldsymbol{v}^*$ is the Laplace transform of $\boldsymbol{v}$. Eq. (7) is of the same form as those derived in Ref. [1] for time harmonic displacements with the constant $i\omega$ replaced by the Laplace transform parameter p. Hence, their general solution can be expressed as

$$v_1^* = \phi_{1,1} + \phi_{2,1} + \psi_{,2}, \qquad v_2^* = \phi_{1,2} + \phi_{2,2} - \psi_{,1}, \qquad v_3^* = e_1\phi_1 + e_2\phi_2 \tag{8}$$

with the transformed displacement potentials ϕ_1, ϕ_2 and ψ satisfying

$$(\nabla^2 - \delta_\alpha^2)\phi_\alpha = 0, \quad \alpha = 1, 2, \text{ no sum on } \alpha, \qquad (\nabla^2 - \delta_3^2)\psi = 0 \tag{9}$$

where

$$e_\beta = -\frac{h(\lambda + 2\mu)}{\kappa\lambda}\left(\delta_\beta^2 - \frac{p^2}{c_1^2}\right), \quad \beta = 1, 2 \tag{10}$$

$$\delta_\beta^2 = \frac{3\kappa^2}{2\alpha_2 h^2}\left[(\alpha_1 + \alpha_2)\frac{p^2}{\bar{\omega}^2} + 1 + (-1)^\beta H\right], \quad \beta = 1, 2, \qquad \delta_3^2 = \frac{p^2}{c_2^2}, \tag{11}$$

$$H = \left\{\left[(\alpha_1 + \alpha_2)\frac{p^2}{\bar{\omega}^2} + 1\right]^2 - 4\alpha_1\alpha_2\frac{p^2}{\bar{\omega}^2}\left(1 + \frac{p^2}{\bar{\omega}^2}\right)\right\}^{1/2}, \quad \alpha_\beta = c_\beta^2/c^2, \ \beta = 1, 2, \tag{12}$$

$$c_1^2 = (\lambda + 2\mu)/\rho, \qquad c_2^2 = \mu/\rho, \qquad c^2 = 4c_2^2(\lambda + \mu)/(\lambda + 2\mu), \tag{13}$$

$$\bar{\omega}^2 = 3\kappa^2 c_1^2/h^2 \tag{14}$$

The Laplace transformed stress resultants $N^*_{\alpha\beta}$, N^*_{33} and R^*_α are given by

$$
\begin{aligned}
N^*_{11}/2\mu &= \sum_{\beta=1}^{2}\left[\left(-\delta_\beta^2+\tfrac{1}{2}\delta_3^2\right)\phi_\beta+\phi_{\beta,11}\right]+\psi_{,12}\\
N^*_{22}/2\mu &= \sum_{\beta=1}^{2}\left[\left(-\delta_\beta^2+\tfrac{1}{2}\delta_3^2\right)\phi_\beta+\phi_{\beta,22}\right]-\psi_{,12}\\
N^*_{12}/2\mu &= \sum_{\beta=1}^{2}\phi_{\beta,12}-\tfrac{1}{2}(\psi_{,11}-\psi_{,22})\\
N^*_{33}/2\mu &= \frac{\kappa}{\nu}\sum_{\beta=1}^{2}\left[-\delta_\beta^2+\tfrac{1}{2}(1-\nu)\delta_3^2\right]\phi_\beta\\
R^*_1/\left(\tfrac{1}{3}\mu h\right) &= \sum_{\beta=1}^{2}e_\beta\phi_{\beta,1}\\
R^*_2/\left(\tfrac{1}{3}\mu h\right) &= \sum_{\beta=1}^{2}e_\beta\phi_{\beta,2}
\end{aligned}
\tag{15}
$$

In Eq. $(15)_4$, ν is Poisson's ratio. Once the potentials ϕ_1, ϕ_2 and ψ are solved from Eq. (9) under appropriate boundary conditions, the Laplace transforms of the displacements and stresses can be obtained from Eqs. (8) and (15).

3. Crack tip fields

By using the standard asymptotic expansion method, it can be shown that the crack tip fields in the Kane–Mindlin plate theory are the same as those of generalized plane strain fracture except that N_{33} differs by $\kappa=\pi/2\sqrt{3}$ which is close to 1. Hence, for a stationary crack subjected to transient loads, the crack tip fields for mode I are

$$v_\alpha=\frac{K_1(t)}{2\mu}\sqrt{\frac{r}{2\pi}}\,\tilde{v}_\alpha(\theta),\qquad v_3=v_3^0+O(r),\tag{16}$$

$$N_{\alpha\beta}=\frac{K_1(t)}{\sqrt{2\pi r}}\tilde{\sigma}_{\alpha\beta}(\theta),\qquad N_{33}=\kappa\nu N_{\alpha\alpha},\qquad R_\alpha=O(1),\tag{17}$$

where $K_1(t)$ is the mode I dynamic stress intensity factor, $\tilde{v}_\alpha(\theta)$ and $\tilde{\sigma}_{\alpha\beta}(\theta)$ are standard angular functions given in fracture mechanics books, e.g. see Ref. [6], and r, θ are polar coordinates such that the crack tip is at $r\to 0$ and the crack faces are $\theta=\pm\pi$.

For a crack steadily growing along the x_1-direction at a speed V, the crack tip fields for mode I are

$$v_\alpha=\frac{K_1(V)}{2\mu}\sqrt{\frac{r}{2\pi}}\,\tilde{V}_\alpha(\theta,V),\qquad v_z=V_z^0+O(r),\tag{18}$$

$$N_{\alpha\beta}=\frac{K_1(V)}{\sqrt{2\pi r}}\tilde{\Sigma}_{\alpha\beta}(\theta,V),\qquad N_{33}=\kappa\nu N_{\alpha\alpha},\qquad R_\alpha=O(1)\tag{19}$$

where $K_1(V)$ is the mode I dynamic stress intensity factor, $\tilde{V}_\alpha(\theta, V)$ and $\tilde{\Sigma}_{\alpha\beta}(\theta, V)$ are angular functions [7] and r, θ are polar coordinates with origin at the moving crack tip.

4. Path-independent integrals

Path-independent integrals play an important role in fracture mechanics [8,9]. Compared to static crack problems, a unified theory of path-independent integrals in dynamic fracture mechanics has not been well developed. A path-independent integral in terms of the Laplace transforms of the field variables is proposed in [10]; it can be related to the Laplace transform of the stress intensity factor. This integral is now extended to include the effects of the plate thickness in the framework of the Kane–Mindlin theory.

The transformed equations of motion are

$$N^*_{\alpha\beta,\beta} = \rho p^2 v^*_\alpha, \qquad R^*_{\alpha,\alpha} - N^*_{33} = \frac{h}{3}\rho p^2 v^*_3 \tag{20}$$

and the transformed stress–strain relations and strain–displacement relations are the same as those given by Eqs. (6) and (5) with field variables there replaced by their Laplace transforms.

Consider the following line integral

$$J^*_{TH} = \int_C \left[(W^* + T^*) n_1 - N^*_{\alpha\beta} n_\beta v^*_{\alpha,1} - \frac{1}{h} R^*_\alpha n_\alpha v^*_{3,1} \right] \mathrm{d}l \tag{21}$$

where C is a simple closed curve, n_α is a unit outward normal to C, $\mathrm{d}l$ is an infinitesimal length on C and

$$W^*(\gamma^*_{\alpha\beta}, \gamma^*_{33}, \gamma^*_\alpha) = \mu\left(\gamma^*_{\alpha\beta}\gamma^*_{\alpha\beta} + (\kappa\gamma^*_{33})^2\right) + \tfrac{1}{2}\lambda(\gamma^*_{\alpha\alpha} + \kappa\gamma^*_{33})^2 + \frac{h^2}{6}\mu\gamma^*_\alpha\gamma^*_\alpha \tag{22}$$

$$\gamma^*_\alpha = \frac{1}{h} v^*_{3,\alpha} = \frac{3}{h^2}\Gamma^*_\alpha \tag{23}$$

$$T^*(v^*_\alpha, v^*_3) = \tfrac{1}{2}\rho p^2 v^*_\alpha v^*_\alpha + \tfrac{1}{6}\rho p^2 (v^*_3)^2 \tag{24}$$

The transformed stress resultants $N^*_{\alpha\beta}$, N^*_{33} and R^*_α are related to W^* by

$$N^*_{\alpha\beta} = \frac{\partial W^*}{\partial \gamma^*_{\alpha\beta}}, \qquad N^*_{33} = \frac{\partial W^*}{\partial \gamma^*_{33}}, \qquad R^*_\alpha = \frac{\partial W^*}{\partial \gamma^*_\alpha} \tag{25}$$

which are equivalent to the transformed stress–strain relations. It should be noted that W^* is not the Laplace transform of the strain energy density of the plate, but has the same form as the strain energy density with the strain resultants replaced by their Laplace transforms. Similarly, T^* is not the Laplace transform of the kinetic energy density of the plate.

By using the Gauss–Green theorem and Eqs. (5), (20), (24) and (25), it can be proved that

$$J^*_{TH} = 0 \tag{26}$$

for any closed curve C enclosing no field singularities.

The conservation law (Eq. (26)) also holds with p replaced by $i\omega$ for steady harmonic motion of a Kane–Mindlin plate with displacements given by

$$v_j(x_1, x_2, t) = \tilde{v}_j(x_1, x_2)\mathrm{e}^{i\omega t} \tag{27}$$

where ω is the frequency and $\tilde{v}_j$ the amplitude.

When J^*_{TH} is evaluated along any contour, Γ, beginning on the lower traction-free crack face, surrounding the crack tip and terminating on the upper traction-free crack face, J^*_{TH} will be independent of the selection of Γ due to its conservative property, cf. Eq. (26). By using the transformed crack tip fields which have the same forms as in Eqs. (16) and (17) with the stress intensity factor $K_\mathrm{I}(t)$ replaced by its Laplace transform $K^*_\mathrm{I}(p)$,the integral J^*_{TH} evaluated on Γ is given by

$$J^*_{TH} = \frac{1-\nu^2}{E}\left[K^*_\mathrm{I}(p)\right]^2 \tag{28}$$

where E and ν are Young's modulus and Poisson's ratio, respectively. For a very thin plate, plane stress singular fields exist in an annular region where $r/a \ll 1$ and $r/h \gg 1$. It can be shown that when $h/a \to 0$, Eqs. (3)–(6) reduce to ones for the plane stress case. Denoting the Laplace transform of the plane stress intensity factor by $K_{\mathrm{I}}^{0*}(p)$, the integral J_{TH}^{*} is now evaluated for Γ within the dominant zone of plane stress singular fields and

$$J_{TH}^{*} = \frac{1}{E}\left[K_{\mathrm{I}}^{0*}(p)\right]^2 \tag{29}$$

Due to the path-independence of J_{TH}^{*}, we obtain for a very thin plate

$$K_{\mathrm{I}}^{*}(p) = \frac{1}{\sqrt{1-\nu^2}} K_{\mathrm{I}}^{0*}(p) \tag{30}$$

The inverse Laplace transform of Eq. (30) gives

$$K_{\mathrm{I}}(t) = \frac{1}{\sqrt{1-\nu^2}} K_{\mathrm{I}}^{0}(t) \tag{31}$$

This enables us to calculate the crack tip dynamic stress intensity factors for a very thin plate from the known plane stress ones. It is noted that the same relation was established in Ref. [3] for a static crack problem.

The path-independent integral J_{TH}^{*} is for stationary cracks subjected to transient loads. For a crack steadily growing along the x_1-direction at speed V, a similar path-independent integral prevails [11]. The same problem in the framework of the Kane–Mindlin Theory will be discussed.

Define

$$\hat{J}_{TH} = \int_C \left[(W+T)n_1 - N_{\alpha\beta} n_\beta v_{\alpha,1} - \frac{1}{h} R_\alpha n_\alpha v_{3,1} \right] \mathrm{d}l \tag{32}$$

where C is a simple closed curve fixed in both size and orientation in the moving coordinate system (x_1, x_2) with the origin always at the crack tip and the crack faces described by $x_2 = 0$, $x_1 < 0$, n_α is the unit outward normal vector of C and $\mathrm{d}l$ is an infinitesimal length on C. In Eq. (32), $W(\gamma_{\alpha\beta}, \gamma_{33}, \gamma_\alpha)$ and $T(\dot{v}, \dot{v}_3)$ are the strain energy density and the kinetic energy density of the plate. W has the same form as W^{*} in Eqs. (22) and (23) and T is given by

$$T(\dot{v}_\alpha, \dot{v}_3) = \tfrac{1}{2}\rho \dot{v}_\alpha \dot{v}_\alpha + \tfrac{1}{6}\rho(\dot{v}_3)^2 \tag{33}$$

The stress resultants can be obtained from W as their Laplace transforms from W^{*} in Eq. (25). Under steady crack growth conditions,

$$(\dot{\ }) = -V\frac{\partial}{\partial x_1}(\ ) \tag{34}$$

By following a procedure similar to that used to prove the conservation of J_{TH}^{*}, it can be shown that

$$\hat{J}_{TH} = 0 \tag{35}$$

for any closed curve C translating with the crack tip and enclosing no field singularities.

When $\hat{J}_{TH}$ is evaluated along any curve Γ translating with the moving coordinate system, beginning on the lower traction-free crack face, surrounding the moving crack tip and terminating on the upper traction-free crack face, $\hat{J}_{TH}$ will be independent of the selection of Γ due to its conservative property, cf. Eq. (35). By using the crack tip fields in Eqs. (18) and (19), we obtain

$$\hat{J}_{TH} = \frac{1-\nu^2}{E} A(V)\left[K_{\mathrm{I}}(V)\right]^2 \tag{36}$$

where $A(V)$ is a universal function identical to that for plane strain deformations [12,13]:

$$A(V)=\frac{(V/c_2)^2(1-V^2/c_1^2)^{1/2}}{(1-\nu)\left[4(1-V^2/c_1^2)^{1/2}(1-V^2/c_2^2)^{1/2}-(2-V^2/c_2^2)^2\right]} \tag{37}$$

For a steadily growing crack, the crack length is usually much larger than the plate thickness. The plane stress singular fields must exist in an annual region where $r/a \ll 1$ and $r/h \gg 1$ if other inplane dimensions are at least an order of magnitude larger than the plate thickness. Denoting the plane stress intensity factor by $K_I^0(V)$, the integral $\hat{J}_{TH}$ is now evaluated for Γ within the dominant zone of plane stress singular fields as

$$\hat{J}_{TH}=\frac{1}{E}B(V)\left[K_I^0(V)\right]^2 \tag{38}$$

where $B(V)$ is obtained from $A(V)$ by replacing Poisson's ratio ν by $\nu/(1+\nu)$ and the dilational wave speed c_1 by $c_1\sqrt{(1-2\nu)}/(1-\nu)$. Due to the path-independence of $\hat{J}_{TH}$, we obtain

$$K_I(V)=\sqrt{\frac{B(V)}{A(V)(1-\nu^2)}}\,K_I^0(V) \tag{39}$$

This relation enables us to calculate the crack tip dynamic stress intensity factors for a thin plate from the known plane stress ones.

5. A semi-infinite crack in an infinite plate

Consider a semi-infinite crack in an infinite plate, with crack surfaces suddenly loaded by a uniform pressure N_0 at $t=0$. The plate is assumed to be initially at rest and stress-free. The corresponding plane strain problem was first solved in Ref. [14]. The stress intensity factor of the plane stress problem can be obtained by replacing Poisson's ratio ν by $\nu/(1+\nu)$, Young's modulus E by $E(1+2\nu)/(1+\nu)^2$ and the dilational wave speed c_1 by $c_1\sqrt{(1-2\nu)}/(1-\nu)$. The result is

$$K_I^0(t)=2\sqrt{2}\left(\frac{1-\nu}{2}\right)^{1/4}\left(\frac{c_R}{c_2}\right)N_0\sqrt{\frac{c_2 t}{\pi}} \tag{40}$$

where c_R is the Rayleigh surface wave speed with ν replaced by $\nu/(1+\nu)$ in the equation determining c_R.

Since the crack is semi-infinite, Eq. (31) holds and the stress intensity factor of the Kane–Mindlin theory is given by

$$K_I(t)=\frac{2\sqrt{2}}{\sqrt{1-\nu^2}}\left(\frac{1-\nu}{2}\right)^{1/4}\left(\frac{c_R}{c_2}\right)N_0\sqrt{\frac{c_2 t}{\pi}} \tag{41}$$

Recall that Eq. (31) holds for traction-free crack faces. However, the stress intensity factor of the present problem is identical to that of a crack problem with traction-free crack faces [15]. The path-independent integral applies to that problem, and hence, equivalently to the present one in determining the stress intensity factor.

6. A finite length crack in an infinite plate

Consider a crack of length $2a$ in an infinite plate. The crack surfaces are loaded suddenly at $t=0$ by a uniform pressure N_0. For $t<0$, the plate is assumed to be at rest and stress free. The boundary conditions of the problem are

$$N_{12}=0, \quad N_{22}=-N_0H(t), \quad R_2=0, \quad |x_1|\le a, \quad x_2=0 \tag{42}$$

$$N_{12}=0, \quad R_2=0, \quad v_2=0, \quad |x_1|>a, \quad x_2=0 \tag{43}$$

$$N_{\alpha\beta}\to 0, \quad N_{33}\to 0, \quad R_\alpha \to 0, \quad x_\alpha x_\alpha \to \infty \tag{44}$$

where $H(t)$ is the Heaviside step function.

By using the Laplace and Fourier transforms, the crack problem can be reduced to the following singular integral equation

$$\int_{-a}^{a}\left\{\frac{1}{\bar{x}_1-x_1}+k(x_1,\bar{x}_1,p)\right\}f^*(\bar{x}_1,p)\,\mathrm{d}\bar{x}_1=-\frac{2\pi(1-\nu^2)}{p}\frac{N_0}{E}, \quad |x_1|\le a \tag{45}$$

where the unknown function $f^*(x_1, p)$ is defined by

$$f^*(x_1,p)=v_1^*(x_1,0)_{,1} \tag{46}$$

and the Fredholm kernel $k(x, \bar{x}, p)$ is

$$k(x,\bar{x},p)=\int_0^\infty\left\{1+\frac{4(1-\nu)}{\delta_3^2}\left[\left(\frac{e_2}{\gamma_1}-\frac{e_1}{\gamma_2}\right)\frac{\left(\xi^2+\frac{1}{2}\delta_3^2\right)^2}{(e_1-e_2)\xi}+\xi\gamma_3\right]\right\}\sin[(x-\bar{x})\xi]\,\mathrm{d}\xi \tag{47}$$

where $\gamma_i=(\xi^2+\delta_i^2)^{1/2}$, $i=1, 2, 3$. The boundary condition (43) implies that

$$\int_{-a}^{a} f^*(x,p)\,\mathrm{d}x=0 \tag{48}$$

The nondimensional form of Eq. (45) is

$$\int_{-1}^{1}\left\{\frac{1}{s-r}+ak(r,s,p)\right\}f^*(s,p)\,\mathrm{d}s=-\frac{2\pi(1-\nu^2)}{p}\frac{N_0}{E}, \quad |r|\le 1 \tag{49}$$

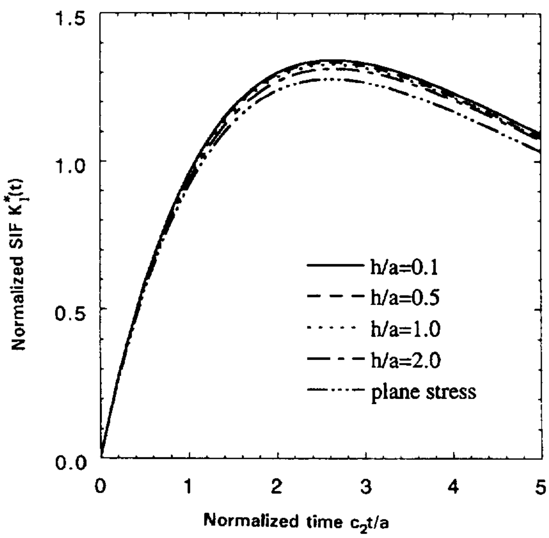

Fig. 1. Dynamic stress intensity factor versus non-dimensional time $c_2 t/a$ for various plate thicknesses ($\nu=0.3$).

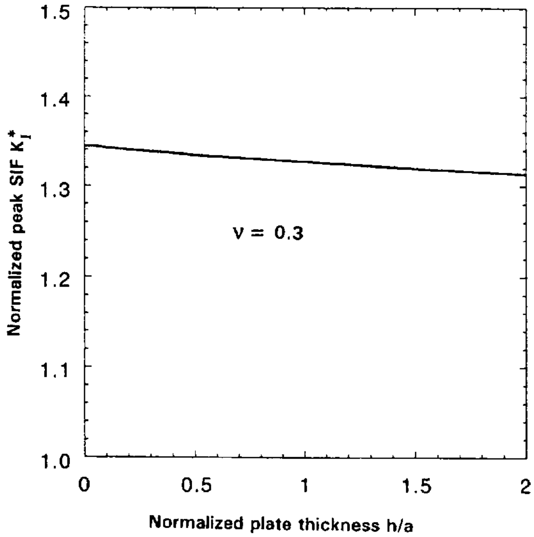

Fig. 2. Peak dynamic stress intensity factor versus non-dimensional plate thickness h/a ($\nu = 0.3$).

where

$$x_1 = ar, \qquad \bar{x}_1 = as. \tag{50}$$

According to the singular integral equation method [16,17] the integral Eq. (49) has a solution of the form

$$f^*(r, p) = \frac{F^*(r, p)}{\sqrt{1 - r^2}} \tag{51}$$

where $F^*(r, p)$ is a continuous bounded function on the interval $[-1, 1]$.

The Laplace transformed stress intensity factor $K_I^*(p)$ is evaluated as

$$K_I^*(p)/N_0\sqrt{\pi a} = -\tfrac{1}{2}F^*(1, p) \tag{52}$$

To obtain the stress intensity factor in the time domain, we need to evaluate the inverse Laplace transform of Eq. (52). Since it is not possible to find an inverse in the closed form, a numerical inversion technique given in Ref. [18] is used. This method has been used in fracture dynamics [6,19].

In the following numerical calculations of the dynamic stress intensity factors (DSIFs), we only consider the case of $\nu = 0.3$. Fig. 1 shows the normalized DSIF versus the nondimensional time $c_2 t/a$ for various values of the plate-thickness parameter h/a. The DSIF is normalized by $N_0/\sqrt{\pi a}$. It can be seen that for each plate thickness, $K_I(t)$ increases with an increase in time, reaches a peak value and then decreases with the increase in time. $K_I(t)$ will approach the static SIF when $t \to \infty$. The peak value is always higher than the steady state value or the static SIF given in Ref. [20] by about 30% and is also higher than the maximum of the plane stress DSIF. Fig. 2 shows the peak DSIF versus the plate thickness parameter h/a. It is observed that the peak value is higher for a thinner plate.

7. Concluding remarks

Dynamic fracture problems of cracked plates are studied by using the Kane–Mindlin plate theory. The Laplace transformed displacements can be represented by three potential functions which satisfy Poisson's equations. For a stationary crack in a very thin plate subjected to transient loads, the crack tip dynamic stress intensity factor (DSIF), $K_I(t)$, can be related to the far field plane stress DSIF, $K_I^0(t)$, by $K_I(t) = K_I^0(t)/\sqrt{1-\nu^2}$. For a crack steadily growing at speed V, the crack tip DSIF, $K_I(V)$, is given by $K_I(V) = \sqrt{B(V)/(A(V)(1-\nu^2))}\,K_I^0(V)$, where $K_I^0(V)$ is the plane stress DSIF. These relations are established by using the path-independent integrals derived for the Kane–Mindlin theory and allow us to obtain the crack tip DSIFs from the known plane stress ones. The results of DSIF for a finite crack in an infinite plate under uniform impact pressure on the crack surfaces show that for each plate thickness, the maximum DSIF is higher than that for the plane stress case.

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