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Higher-order shear and normal deformable theory for functionally graded incompressible linear elastic plates

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Dedicated to my friend and colleague Professor M.W. Hyer on his 65th birthday

Abstract

We use the principle of virtual work to derive a higher-order shear and normal deformable theory for a plate comprised of a linear elastic incompressible anisotropic material. The theory does not use a shear correction factor and employs three components of displacement and the hydrostatic pressure as independent variables. For a Kth order plate theory, a set of $4(K+1)$ coupled equations need to be solved for the $(K+1)$ pressures and the $3(K+1)$ displacements defined on the reference surface of the plate.

Equations for free vibrations of a plate are derived, and equations for the determination of frequencies and the corresponding mode shapes of a simply supported rectangular plate are given.

Keywords: Plate theory; Rubberlike materials; Free vibrations; Functionally graded material

1. Introduction

Because of the increasing interest in rubberlike materials and elastomers and their use in automotive and aerospace industries we develop here a higher-order shear and normal deformable theory for plates made of incompressible linear elastic materials. Only isochoric or volume preserving deformations are admissible in an incompressible material. Accordingly, the constitutive relation for an incompressible material involves a hydrostatic pressure that cannot be determined from the strain field but is found by solving equations governing deformations of the body and the prescribed initial and boundary conditions. The pressure field can be determined uniquely only if normal surface tractions are prescribed on a part of the boundary of the body. Thus for a plate made of an incompressible material we need not only equations to find the displacement field but also the pressure field.

Aimmanee and Batra [1] have recently given an analytical solution for free vibrations of a simply supported rectangular plate made of an incompressible linear elastic material. For a thick plate their results show that the displacement and the pressure fields do not vary linearly through the plate thickness. Also, for a thick plate frequencies of some in-plane modes of vibration with the lateral deflection identically zero are lower than those of the out-of-plane modes of vibration with non-vanishing lateral deflections. We develop here a plate theory in which the three displacement components and the pressure are expanded in Taylor series in the thickness coordinate, z, and terms of the same degree in z are retained in their expansions. Since both transverse shear and transverse normal strains are considered, we call the theory shear and normal deformable. The plate theory is called Kth order if terms of order z^K are kept in the Taylor series expansions in z of displacements and the pressure field. The order of the theory suitable for a plate depends upon the ratio, R, of the plate thickness to the characteristic in-plane dimension and which aspects of the 3-dimensional (3-D) deformations are to be well approximated; a small value of K suffices for a thin plate with $R \ll 1$.

Batra and Vidoli [2] followed Mindlin and Medick [3] in using Legendre polynomials in z as basis functions to

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derive a mixed higher-order shear and normal deformable plate theory (HOSNDPT) for piezoelectric plates from a mixed variational principle [4]. Subsequently, Batra et al. [5] employed the Reissner–Hellinger mixed principle to deduce a mixed HOSNDPT and also a compatible HOSNDPT. Whereas in the former independent expansions for displacements and stresses are presumed, in the latter only displacements are expanded. Strains deduced from the displacement field and the pertinent constitutive relation are used to find stresses. Here we employ the principle of virtual work to derive a compatible HOSNDPT for a plate made of an incompressible anisotropic linear elastic inhomogeneous or functionally graded material. We do not require that the transverse normal and/or the transverse shear strains must vanish on the top and the bottom surfaces of a plate. Thus the effect of tangential tractions applied on these surfaces can be studied.

There are numerous papers on plate theories. However, they assume the plate material to be compressible. Herrmann [6] reorganized equations of linear elasticity so that they are applicable to both compressible and incompressible materials; these also involve the three components of displacement and the pressure as unknowns. Other works that consider transverse normal and transverse shear strains include those of Mindlin and Medick [3], Vlasov [7], Lo et al. [8], Kant [9], Hanna and Leissa [10], Reddy [11], Lee and Yu [12], and Lee et al. [13]. This list is by no means complete since the number of papers on plate theories is too large to be included here.

2. Formulation of the problem

We use rectangular Cartesian coordinates to describe infinitesimal deformations of a plate made of an incompressible linear elastic material. Let the x_1x_2-plane coincide with the mid-surface of the plate and the x_3-axis be along the thickness direction. With h equaling the plate thickness, we normalize lengths by $h/2$ so that $x_3 = +1$ and -1 at points on the top and the bottom surfaces, S^+ and S^-, respectively, of the plate.

Equations governing 3-D infinitesimal deformations of a linear elastic incompressible body are

$$\sigma_{ij,j} + \rho b_i = \rho \ddot{u}_i \quad \text{in } \Omega, \; i = 1,2,3, \tag{1}$$

$$u_{i,i} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\sigma_{ij} n_j = \bar{t}_i \quad \text{on } \partial_t \Omega, \; u_i = \bar{u}_i \; \text{ on } \partial_u \Omega, \tag{3}$$

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}) \quad \text{in } \Omega, \tag{4}$$

$$\sigma_{ij} = -p\delta_{ij} + \hat{\sigma}_{ij} \quad \text{in } \Omega, \tag{5}$$

$$\hat{\sigma}_{ij} = C_{ijkl} e_{kl} \quad \text{in } \Omega, \; C_{ijkl} = C_{jikl} = C_{ijlk}, \tag{6}$$

$$e_{kl} = (u_{k,l} + u_{l,k})/2 \quad \text{in } \Omega. \tag{7}$$

Here and below, ρ is the mass density, σ_{ij} the stress tensor, b_i the body force per unit mass, a superimposed dot denotes partial differentiation with respect to time t, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$, and Ω is the 3-D region occupied by the plate. Furthermore, u_i is the displacement component along the x_i-direction, $\mathbf{n}$ an outward unit normal to the boundary $\partial\Omega$ of Ω, $\bar{t}_i$ the prescribed traction at points on $\partial_t\Omega$, $\bar{u}_i$ the prescribed displacement at points on $\partial_u\Omega$, u_i^0 the prescribed initial displacement, and $\dot{u}_i^0$ the assigned initial velocity field. A repeated index implies summation over the range of the index. Eq. (1) expresses the balance of linear momentum, (2) the continuity equation or the balance of mass, $(3)_1$ and $(3)_2$ boundary conditions, and $(4)_1$ and $(4)_2$ are initial conditions. Eq. (5) is the constitutive relation in which δ_{ij} is the Kronecker delta, and the hydrostatic pressure p cannot be determined from the deformation field but can be found by solving the system of partial differential equations (1) and boundary conditions (3). The pressure p is uniquely determined only if normal surface tractions are prescribed on a part of the boundary. The part $\hat{\sigma}_{ij}$ of the stress tensor in Eq. (5) is a linear function of the infinitesimal strain e_{ij}. The number of independent components of the elasticity tensor C_{ijkl} equals 1, 4, 8 and 20 for an isotropic, transversely isotropic, orthotropic, and anisotropic incompressible material respectively. Expressions for C_{ijkl} in terms of more familiar elastic constants are given, e.g., in [14]. For an inhomogeneous body (e.g. functionally graded) elasticities C_{ijkl} vary continuously throughout the body. The infinitesimal strain e_{ij} is related to the displacement gradients by Eq. (7).

For a plate usually surface tractions are prescribed on its top, S^+, and bottom, S^-, surfaces. However, displacements and/or surface tractions may be prescribed on edge surfaces of the plate.

In the absence of body forces, Eqs. (1), (2) and (5)–(7) imply that the hydrostatic pressure p is a harmonic function for a homogeneous and isotropic body. That is, it satisfies the Laplace equation.

3. Derivation of the higher-order plate theory

Let $L_0(z), L_1(z), L_2(z), \ldots$ be orthonormal Legendre polynomials satisfying

$$\int_{-1}^{1} L_a(z) L_b(z)\, \mathrm{d}z = \delta_{ab}, \quad a, b = 0, 1, 2, \ldots, K. \tag{8}$$

Note that

$$L_0(z) = \frac{1}{\sqrt{2}}, \quad L_1(z) = \sqrt{\frac{3}{2}} z, \quad L_2(z) = \sqrt{\frac{5}{2}} \left(\frac{3z^2}{2} - \frac{1}{2} \right),$$
$$L_3(z) = \sqrt{\frac{7}{2}} \left(-3\frac{z}{2} + \frac{5}{2} z^3 \right), \ldots . \tag{9}$$

The basis functions $L_0(z), L_1(z), \ldots, L_K(z)$ are equivalent to $1, z, z^2, \ldots, z^K$, and are, alternatively, even and odd functions of z. An advantage of using orthonormal basis functions is that the algebraic work is reduced. Henceforth we set $x_3 = z$, and

$$u_i(x_1, x_2, z, t) = v_\alpha(x_1, x_2, z, t)\delta_{i\alpha} + w(x_1, x_2, z, t)\delta_{i3},\quad \alpha = 1, 2. \tag{10}$$

That is, v_α denotes the in-plane components of displacement, and w the lateral deflection of a point in the plate. Let

$$v_\alpha(x_1, x_2, z, t) = L_a(z)v^a_\alpha(x_1, x_2, t), \quad a = 0, 1, 2, \ldots, K,$$

$$w(x_1, x_2, z, t) = L_a(z)w^a(x_1, x_2, t), \tag{11}$$

$$p(x_1, x_2, z, t) = L_a(z)p^a(x_1, x_2, t).$$

Here and below the repeated index a is summed even if it appears as a subscript and a superscript. Noting that $L'_a(z) = \mathrm{d}L_a(z)/\mathrm{d}z$ can be expressed as a linear combination of $L_0, L_1, L_2, \ldots, L_K$, we write it as

$$L'_a(z) = D_{ab}L_b(z). \tag{12}$$

For $K = 7$, the 8×8 matrix $\mathbf{D}$ is given by

$$[D] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{7} & 0 & \sqrt{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\sqrt{3} & 0 & 3\sqrt{7} & 0 & 0 & 0 & 0 \\ \sqrt{11} & 0 & \sqrt{55} & 0 & 3\sqrt{13} & 0 & 0 & 0 \\ 0 & \sqrt{39} & 0 & \sqrt{91} & 0 & \sqrt{143} & 0 & 0 \\ \sqrt{15} & 0 & 5\sqrt{3} & 0 & 3\sqrt{15} & 0 & \sqrt{195} & 0 \end{bmatrix}. \tag{13}$$

Note that all elements of the first row and the last column of the matrix D are zeroes. Eqs. (11) and (12) give

$$v_{\alpha,\beta} = L_a v^a_{\alpha,\beta}, \quad v_{\alpha,z} = D_{ab}L_b v^a_\alpha,$$

$$w_{,\alpha} = L_a w^a_{,\alpha}, \quad w_{,z} = D_{ab}L_b w^a. \tag{14}$$

Thus

$$e_{ij} = L_a[\tfrac{1}{2}(v^a_{\alpha,\beta} + v^a_{\beta,\alpha})\delta_{i\alpha}\delta_{j\beta} + \tfrac{1}{2}(\delta_{i\alpha}\delta_{j3} + \delta_{i3}\delta_{j\alpha}) \times (D_{ba}v^b_\alpha + w^a_{,\alpha}) + \delta_{i3}\delta_{j3}D_{ba}w^b], \tag{15}$$

$$\equiv L_a e^a_{ij}. \tag{16}$$

Here e^a_{ij} can be thought of as the strain derived from the ath-order displacement field, and terms on the right-hand side of Eq. (11) as Taylor series expansions of v_α, w and p in the z-direction. Recall that for an incompressible material the pressure p is to be determined as a part of the solution of the problem. The value of K or equivalently the number of terms to be retained in Eq. (11) depends upon R, and which aspects of the 3-D deformations in the plate theory are to be approximated well. The plate theory is called higher-order if $K \geqslant 2$.

4. Derivation of plate theory equations

4.1. Balance laws

Let η_i be a smooth virtual displacement field defined on Ω that vanishes on $\partial_u\Omega$, and λ be a smooth scalar field defined on Ω. Taking the inner product of both sides of Eq. (1) with $\boldsymbol{\eta}$, multiplying both sides of Eq. (2) with a Lagrange multiplier function λ, and integrating the resulting equations over Ω, we obtain

$$\int_S \mathrm{d}A \int_{-1}^{1} \eta_i\sigma_{ij,j}\,\mathrm{d}z + \int_S \mathrm{d}A \int_{-1}^{1} \rho b_i\eta_i\,\mathrm{d}z = \int_S \mathrm{d}A \int_{-1}^{1} \rho\ddot{u}_i\eta_i\,\mathrm{d}z, \tag{17}$$

$$\int_S \mathrm{d}A \int_{-1}^{1} \lambda u_{i,i}\,\mathrm{d}z = 0, \tag{18}$$

where the area integration is over the midsurface S of the plate. We assume that

$$\eta_i(x_1, x_2, z) = L_a(z)\eta^a_i(x_1, x_2),$$

$$\lambda(x_1, x_2, z) = L_a(z)\lambda^a(x_1, x_2), \tag{19}$$

substitute these into Eqs. (17) and (18), define

$$M^a_{\alpha\beta} = \int_{-1}^{1} \sigma_{\alpha\beta}L_a\,\mathrm{d}z, \quad T^a_i = \int_{-1}^{1} \sigma_{i3}L_a\,\mathrm{d}z,$$

$$R_{ab} = \int_{-1}^{1} \rho L_a L_b\,\mathrm{d}z,$$

$$b^a_i = \int_{-1}^{1} \rho L_a b_i\,\mathrm{d}z,$$

$$B^a_i = L_a(1)\sigma_{i3}(x_1, x_2, 1) - L_a(-1)\sigma_{i3}(x_1, x_2, -1), \tag{20}$$

and obtain

$$\int_S \eta^a_\alpha M^a_{\alpha\beta,\beta}\,\mathrm{d}A + \int_S \eta^a_3 T^a_{\alpha,\alpha}\,\mathrm{d}A + \int_S \eta^a_i(b^a_i + B^a_i - D_{ab}T^b_i)\,\mathrm{d}A = R_{ab}\int_S \eta^a_i\ddot{u}^b_i\,\mathrm{d}A, \tag{21}$$

$$\int_S \lambda^a(v^a_{\alpha,\alpha} + D_{ca}w^c)\,\mathrm{d}A = 0. \tag{22}$$

Here $M^0_{\alpha\beta}$ are the in-plane forces, $M^1_{\alpha\beta}$ the out-of-plane moments, and $M^a_{\alpha\beta}$, $a \geqslant 2$ are the higher order out-of-plane moments. Similarly T^0_i equal the resultant transverse forces, T^1_i the first-order moment of the transverse forces, and $T^a_i, a \geqslant 2$ the higher-order moment of transverse forces. R_{00} equals the mass per unit area of the plate, R_{11} the rotary inertia, and R_{ab}, $a, b \geqslant 2$ the higher-order inertia. Furthermore b^a_i are ath-order moments of the body force about the mid-surface of the plate. In particular, b^0_i equals the body force per unit surface area, and b^1_i the moment per unit area of the body force. B^a_i are the ath-order moments of surface tractions applied on the top and the bottom surfaces of the plate. For equal and opposite surface tractions prescribed on the top and the bottom surfaces of the plate, $B^0_i = B^2_i = B^4_i = \cdots = 0$, but B^1_i, $B^3_i \ldots$ are non-zero. Thus squeezing of a plate and/or its deformations due to equal and opposite normal tractions on the top and the bottom surfaces of the plate can be analyzed. Furthermore, the transverse shear strains are not required to vanish on the top and the bottom surfaces of the plate. Thus the effect of tangential tractions applied on these surfaces can be considered. Since Eqs. (21) and (22) must hold for all choices of η_i^a and λ^a, we get

$$M^a_{\alpha\beta,\beta} - D_{ab}T^b_\alpha + b^a_\alpha + B^a_\alpha = R_{ab}\ddot{v}^b_\alpha, \tag{23}$$

$$T^a_{\alpha,\alpha} - D_{ab}T^b_3 + b^a_3 + B^a_3 = R_{ab}\ddot{w}^b, \tag{24}$$

$$v^a_{\alpha,\alpha} + D_{ca}w^c = 0. \tag{25}$$

Substitution for σ_{ij} from Eq. (5) and for p from Eq. $(11)_3$ into Eqs. $(20)_1$ and $(20)_2$ gives

$$M^a_{\alpha\beta} = -p^a\delta_{\alpha\beta} + \hat{M}^a_{\alpha\beta}, \quad \hat{M}^a_{\alpha\beta} = \int_{-1}^{1} L_a\hat{\sigma}_{\alpha\beta}\,\mathrm{d}z,$$

$$T^a_3 = -p^a + \hat{T}^a_3, \;\; \hat{T}^a_3 = \int_{-1}^{1} L_a\hat{\sigma}_{33}\,\mathrm{d}z. \tag{26}$$

Thus Eqs. (23) and (24) simplify to

$$\hat{M}^a_{\alpha\beta,\beta} - p^a_{,\alpha} - D_{ab}T^b_\alpha + b^a_\alpha + B^a_\alpha = R_{ab}\ddot{v}^b_\alpha, \tag{27}$$

$$T^a_{\alpha,\alpha} + D_{ab}p^b - D_{ab}\hat{T}^b_3 + b^a_3 + B^a_3 = R_{ab}\ddot{w}^b. \tag{28}$$

Eqs. (27), (28) and (25) are the balance laws for the plate theory, and subject to appropriate initial and boundary conditions are to be solved simultaneously for v^a, w^a and p^a. These need to be supplemented by constitutive relations, and initial and boundary conditions which are discussed below.

4.2. Constitutive relations

Substitution for $\hat{\boldsymbol{\sigma}}$ from Eq. (6) into Eqs. $(26)_2$, $(26)_4$ and $(20)_2$, and then for e_{ij} from Eq. (15) into the resulting equations give

$$\hat{M}^a_{\alpha\beta} = \left(\int_{-1}^{1} L_a C_{\alpha\beta kl} L_b\,\mathrm{d}z\right) e^b_{kl},$$

$$\hat{T}^a_3 = \left(\int_{-1}^{1} C_{33kl} L_a L_b\,\mathrm{d}z\right) e^b_{kl},$$

$$\hat{T}^a_\alpha = \left(\int_{-1}^{1} C_{\alpha 3kl} L_a L_b\,\mathrm{d}z\right) e^b_{kl}. \tag{29}$$

For a homogeneous body the elasticities C_{ijkl} and the mass density ρ are constants. Eqs. $(20)_3$ and (29) simplify to

$$R_{ab} = \rho\delta_{ab}, \quad \hat{M}^a_{\alpha\beta} = C_{\alpha\beta kl}e^a_{kl}, \quad \hat{T}^a_3 = C_{33kl}e^a_{kl},$$
$$T^a_\alpha = C_{\alpha 3kl}e^a_{kl}. \tag{30}$$

Because of the use of the orthonormal Legendre polynomials as basis functions the rotary inertia R_{11} equals ρ in our formulation.

For an inhomogeneous (often called functionally graded) body, the elasticities and the mass density may be expressed as

$$C_{ijkl} = L_a(z)C^a_{ijkl}(x_1, x_2), \quad \rho = L_a(z)\rho^a(x_1, x_2). \tag{31}$$

Vel and Batra [15–17] and Lee and Yu [12] have employed expansions (31) and the assumptions of C^a_{ijkl} and ρ^a being independent of x_1 and x_2 to solve problems for plates made of functionally graded materials with material properties varying only in the z-direction. For known spatial variations of C_{ijkl} and ρ the definite integrals in Eqs. $(20)_3$ and (29) can be evaluated by using an appropriate quadrature rule as has been done, for example, by Qian and Batra [18] and Gilhooley et al. [19] for plates made of a compressible material. The variation in the z-direction of the mass density and the material elasticities may influence the order K of the plate theory to be used. Whereas Refs. [12,15–19] assumed that material properties vary only in the thickness direction, Qian and Batra [20] determined the variation of material properties in the x_1x_2-plane so as to optimize the lowest frequency of free vibration of the plate. Batra and Jin [21] have found the gradation in material moduli along the thickness direction by varying the fiber orientation angle that optimizes the fundamental frequency. Eqs. (29), (31) and $(20)_3$ give

$$\hat{M}^a_{\alpha\beta} = A_{abd}C^d_{\alpha\beta kl}e^b_{kl}, \quad T^a_\alpha = A_{abd}C^d_{\alpha 3kl}e^b_{kl},$$
$$\hat{T}^a_3 = A_{abd}C^d_{33kl}e^b_{kl}, \quad A_{abd} = \int_{-1}^{1} L_aL_bL_d\,\mathrm{d}z,$$
$$R_{ab} = A_{abd}\rho^d. \tag{32}$$

The quantities A_{abc} can be evaluated exactly since they involve simple polynomials in z. The number of terms retained in expansions (31) need not equal the order K of the plate theory.

4.3. Boundary conditions

The traction boundary conditions (3) on the top and the bottom surfaces of the plate have been incorporated in Eqs. (23) and (24). If surface tractions $\bar{t}_i$ are prescribed on an edge surface of the plate with the outward unit normal $\mathbf{n}$, then the corresponding boundary conditions for the plate theory are

$$\int_{-1}^{1} L_a\sigma_{ij}n_j\,\mathrm{d}z = \int_{-1}^{1} L_a\bar{t}_i\,\mathrm{d}z. \tag{33}$$

Assuming that $n_j = \delta_{j\alpha}n_\alpha$, i.e., the outward unit normal to the edge surface of the plate is in the x_1x_2-plane, then $n_3 = 0$, and Eq. (33) becomes

$$M^a_{\alpha\beta}n_\beta = \int_{-1}^{1} L_a\bar{t}_\alpha\,\mathrm{d}z, \quad T^a_3 = \int_{-1}^{1} L_a\bar{t}_3\,\mathrm{d}z. \tag{34}$$

In particular, if the edge surface $x_1 =$ constant is traction free, then $n_i = \delta_{1i}$, and boundary conditions (34) on it become

$$M^a_{\alpha 1} = 0, \quad T^a_3 = 0, \quad \alpha = 1, 2; \; a = 0, 1, 2, \ldots, K. \tag{35}$$

From the displacement boundary conditions $(3)_2$ we conclude that

$$v^a_\alpha = \int_{-1}^{1} L_a \bar{u}_\alpha \, dz, \quad w^a = \int_{-1}^{1} L_a \bar{u}_3 \, dz. \tag{36}$$

Thus at a clamped edge,

$$v^a_\alpha = 0, \quad w^a = 0. \tag{37}$$

Corresponding to the boundary conditions $w = 0$, $\sigma_{\alpha\beta} n_\beta = 0$ at a simply supported edge, we have

$$w = 0, \quad M_{\alpha\beta} n_\beta = 0, \tag{38}$$

on a simply supported edge. There is general agreement among practioners about boundary conditions at a free edge and at a clamped edge. However, boundary conditions (38) at a simply supported edge are not universally accepted.

4.4. Initial conditions

Substitution from Eqs. (10) and (11) into Eqs. $(4)_1$ and $(4)_2$, multiplication of both sides of the resulting equations with $L_a(z)$, and integration with respect to z on $[-1, 1]$ yield

$$\begin{aligned}
v^a_\alpha(x_1, x_2, 0) &= \int_{-1}^{1} u^0_\alpha(x_1, x_2, z) L_a(z) \, dz, \\
w^a(x_1, x_2, 0) &= \int_{-1}^{1} u^0_3(x_1, x_2, z) L_a(z) \, dz, \\
\dot{v}^a_\alpha(x_1, x_2, 0) &= \int_{-1}^{1} \dot{u}^0_\alpha(x_1, x_2, z) L_a(z) \, dz, \\
\dot{w}^a(x_1, x_2, 0) &= \int_{-1}^{1} \dot{u}^0_3(x_1, x_2, z) L_a(z) \, dz.
\end{aligned} \tag{39}$$

4.5. Special cases

We presume that the plate is made of a homogeneous and isotropic material. Thus

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{40}$$

where μ is the shear modulus. There is only one material parameter for an isotropic incompressible linear elastic material.

For $K = 0$, Eqs. (11), (25), (27), (28) and (30) give

$$\begin{aligned}
v_\alpha(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} v^0_\alpha(x_1, x_2, t), \quad \alpha = 1, 2, \\
w(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} w^0(x_1, x_2, t), \\
p(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} p^0(x_1, x_2, t);
\end{aligned} \tag{41}$$

$$\begin{aligned}
& v^0_{\alpha,\alpha} = 0, \\
& \hat{M}^0_{\alpha\beta,\beta} - p^0_{,\alpha} + b^0_\alpha + B^0_\alpha = \rho \ddot{v}^0_\alpha, \\
& T^0_{\alpha,\alpha} + b^0_3 + B^0_3 = \rho \ddot{w}^0;
\end{aligned} \tag{42}$$

$$\begin{aligned}
& \hat{M}^0_{\alpha\beta} = 2\mu e^0_{\alpha\beta}, \quad \hat{T}^0_3 = 2\mu e^0_{33}, \quad \hat{T}^0_\alpha = 2\mu e^0_{3\alpha}, \\
& e^0_{\alpha\beta} = \tfrac{1}{2}(v^0_{\alpha,\beta} + v^0_{\beta,\alpha}), \quad e^0_{33} = 0, \quad e^0_{\alpha 3} = \tfrac{1}{2} w^0_{,\alpha}.
\end{aligned} \tag{43}$$

Substitution from Eqs. (43) into $(42)_{2,3}$ and recalling Eq. $(42)_1$ give

$$\begin{aligned}
& \mu v^0_{\alpha,\beta\beta} - p^0_{,\alpha} + b^0_\alpha + B^0_\alpha = \rho \ddot{v}^0_\alpha, \\
& \mu w^0_{,\alpha\alpha} + b^0_3 + B^0_3 = \rho \ddot{w}^0.
\end{aligned} \tag{44}$$

In this theory, the plate thickness remains unaltered and deformations associated with the bending of the plate are ignored. The incompressibility condition requires that the area of each infinitesimal element of plate's midsurface remain unchanged as signified by Eq. $(42)_1$. The pressure field is independent of the z-coordinate; its variation on plate's midsurface is embedded in Eq. $(44)_1$. Eqs. $(44)_1$ and $(44)_2$ governing the in-plane and the transverse displacements of a point are uncoupled. Eq. $(44)_2$ coincides with the equation governing transverse deflections of a membrane. Eqs. $(42)_1$ and $(44)_1$ determine the in-plane stretching of the membrane and the hydrostatic pressure in the plate. Needless to say this theory will give very approximate results for a plate, and the approximation improves with a decrease in the ratio, R, of the plate thickness to the largest in-plane dimension.

For $K = 1$, Eqs. (11), (25), (27), (28) and (30) yield

$$\begin{aligned}
v_\alpha(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} v^0_\alpha(x_1, x_2, t) + \sqrt{\frac{3}{2}} z v^1_\alpha(x_1, x_2, t), \\
w(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} w^0(x_1, x_2, t) + \sqrt{\frac{3}{2}} z w^1(x_1, x_2, t), \\
p(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} p^0(x_1, x_2, t) + \sqrt{\frac{3}{2}} z p^1(x_1, x_2, t);
\end{aligned} \tag{45}$$

$$\begin{aligned}
& v^0_{\alpha,\alpha} + \sqrt{3} w^1 = 0, \\
& \hat{M}^0_{\alpha\beta,\beta} - p^0_{,\alpha} + b^0_\alpha + B^0_\alpha = \rho \ddot{v}^0_\alpha, \\
& T^0_{\alpha,\alpha} + b^0_3 + B^0_3 = \rho \ddot{w}^0;
\end{aligned} \tag{46}$$

$$\begin{aligned}
& v^1_{\alpha,\alpha} = 0, \\
& \hat{M}^1_{\alpha\beta,\beta} - p^1_{,\alpha} - \sqrt{3} T^0_\alpha + b^1_\alpha + B^1_\alpha = \rho \ddot{v}^1_\alpha, \\
& T^1_{\alpha,\alpha} + \sqrt{3} p^0 - \sqrt{3} \hat{T}^0_3 + b^1_3 + B^1_3 = \rho \ddot{w}^1;
\end{aligned} \tag{47}$$

$$\hat{M}^a_{\alpha\beta} = 2\mu e^a_{\alpha\beta}, \quad \hat{T}^a_3 = 2\mu e^a_{33}, \quad T^a_\alpha = 2\mu e^a_{3\alpha}, \quad a = 0, 1;$$

$$e^a_{\alpha\beta} = \tfrac{1}{2}(v^a_{\alpha,\beta} + v^a_{\beta,\alpha}), \quad e^0_{33} = \sqrt{3}w^1, \quad e^0_{\alpha 3} = \tfrac{1}{2}(w^0_{,\alpha} + \sqrt{3}v^1_\alpha), \tag{48}$$

$$e^1_{33} = 0, \quad 2e^1_{\alpha 3} = w^1_{,\alpha}.$$

This theory generalizes the first-order shear deformation theory in that it also considers changes in the plate thickness or equivalently the term w^1 representing transverse normal strains. The in-plane displacements, the transverse deflection and the pressure field are assumed to vary linearly through the plate thickness.

The first order transverse displacement w^1 affects the zeroth-order in-plane displacement v^0_α through Eqs. $(46)_1$ and $(47)_2$. Fields p^0, v^0_α, w^0 for the first-order shear and normal deformable plate theory are not necessarily the same as those for the zeroth-order plate theory.

Note that the partial differential equations for $v^0_\alpha, w^0, v^1_\alpha$ and w^1 are second-order. Thus no special techniques are needed to find an approximate solution of the initial-boundary-value problems defined on the mid-surface S of the plate.

The constraints $w^1 = 0$, $v^1_1 = -\partial w^0/\partial x_1$, $v^1_2 = -\partial w^0/\partial x_2$, $v^0_\alpha = 0$ will give the Kirchhoff plate theory. The constraint $w^1 = 0$ will reduce the present first-order shear and normal deformable plate theory to the first-order shear deformation theory.

For $K = 3$,

$$\begin{aligned} v_\alpha(x_1, x_2, z, t) &= \frac{1}{\sqrt{2}} v^0_\alpha(x_1, x_2, t) + \sqrt{\frac{3}{2}} z v^1_\alpha(x_1, x_2, t) \\ &+ \sqrt{\frac{5}{2}} \frac{3z^2 - 1}{2} v^2_\alpha(x_1, x_2, t) \\ &+ \sqrt{\frac{7}{2}} \left(\frac{-3z + 5z^3}{2}\right) v^3_\alpha(x_1, x_2, t), \end{aligned} \tag{49}$$

and similar expressions hold for w and p. Eqs. $(46)_{2,3}$ and $(47)_{2,3}$ are supplemented with the following equations.

$$\begin{aligned} &v^0_{\alpha,\alpha} + \sqrt{3}w^1 + \sqrt{7}w^3 = 0, \\ &v^1_{\alpha,\alpha} + \sqrt{15}w^2 = 0, \\ &v^2_{\alpha,\alpha} + \sqrt{35}w^3 = 0, \\ &v^3_{\alpha,\alpha} = 0; \end{aligned} \tag{50}$$

$$\begin{aligned} &\hat{M}^2_{\alpha\beta,\beta} - p^2_{,\alpha} - \sqrt{15}T^1_\alpha + b^2_\alpha + B^2_\alpha = \rho \ddot{v}^2_\alpha, \\ &T^2_{\alpha,\alpha} + \sqrt{15}p^1 - \sqrt{15}\hat{T}^1_3 + b^2_3 + B^2_3 = \rho \ddot{w}^2; \end{aligned} \tag{51}$$

$$\begin{aligned} &\hat{M}^3_{\alpha\beta,\beta} - p^3_{,\alpha} - (\sqrt{7}T^0_\alpha + \sqrt{35}T^2_\alpha) + b^3_\alpha + B^3_\alpha = \rho \ddot{v}^3_\alpha, \\ &T^3_{\alpha,\alpha} + (\sqrt{7}p^0 + \sqrt{35}p^2) - (\sqrt{7}\hat{T}^0_3 + \sqrt{35}\hat{T}^2_3) + b^3_3 + B^3_3 = \rho \ddot{w}^3; \end{aligned} \tag{52}$$

$$\begin{aligned} &\hat{M}^a_{\alpha\beta} = 2\mu e^a_{\alpha\beta},\ T^a_\alpha = 2\mu e^a_{\alpha 3},\ \hat{T}^a_3 = 2\mu \hat{e}^a_{33},\ a = 0, 1, 2, 3; \\ &e^a_{\alpha\beta} = (v^a_{\alpha,\beta} + v^a_{\beta,\alpha})/2, \\ &e^0_{33} = \sqrt{3}w^1 + \sqrt{7}w^3,\ e^1_{33} = \sqrt{15}w^2,\ e^2_{33} = \sqrt{35}w^3,\ e^3_{33} = 0, \\ &2e^0_{\alpha 3} = \sqrt{3}v^1_\alpha + \sqrt{7}v^3_\alpha + w^0_{,\alpha}, \\ &2e^1_{\alpha 3} = w^1_{,\alpha} + \sqrt{15}v^2, 2e^2_{\alpha 3} = w^2_{,\alpha} + \sqrt{35}w^3, 2e^3_{\alpha 3} = w^3_{,\alpha}. \end{aligned} \tag{53}$$

Thus for a given value of K, $(K+1)$ initial-boundary-value problems need to be solved. One way to ascertain the appropriate value of K and hence of the order of the plate theory is to find the difference between solutions for two successive values of K and terminate the solution process when this difference is less than the acceptable tolerance.

5. Free vibrations

When studying free vibrations of a plate, we take $\bar{t}_i = 0$ on $\partial_t \Omega$, and $\mathbf{b} = \mathbf{0}$. Thus Eqs. (27), (28) and (25) become

$$\begin{aligned} &\hat{M}^a_{\alpha\beta,\beta} - p^a_{,\alpha} - D_{ab}T^b_\alpha = R_{ab}\ddot{v}^b_\alpha, \\ &T^a_{\alpha,\alpha} + D_{ab}p^b - D_{ab}\hat{T}^b_3 = R_{ab}\ddot{w}^b, \\ &v^a_{\alpha,\alpha} + D_{ca}w^c = 0. \end{aligned} \tag{54}$$

Substitution for $\hat{M}^a_{\alpha\beta}$, $\hat{T}^a_3$ and T^a_α from Eq. (30) into Eq. (54) gives

$$\begin{aligned} &(C_{\alpha\beta kl}e^a_{kl})_{,\beta} - p^a_{,\alpha} - D_{ab}C_{\alpha 3kl}e^b_{kl} = R_{ab}\ddot{v}^b_\alpha, \\ &(C_{\alpha 3kl}e^a_{kl})_{,\alpha} + D_{ab}p^b - D_{ab}C_{33kl}e^b_{kl} = R_{ab}\ddot{w}^b, \\ &v^a_{\alpha,\alpha} + D_{ca}w^c = 0. \end{aligned} \tag{55}$$

Henceforth we assume that the plate is made of a homogeneous and isotropic material. Furthermore, we seek solutions of Eq. (55) of the following form:

$$\begin{aligned} &v^a_\alpha(x_1, x_2, t) = e^{i\omega t} V^a_\alpha(x_1, x_2), \\ &w^a(x_1, x_2, t) = e^{i\omega t} W^a(x_1, x_2), \\ &p^a(x_1, x_2, t) = e^{i\omega t} P^a(x_1, x_2). \end{aligned} \tag{56}$$

We substitute in Eq. (55) for e^a_{kl} from Eqs. (15) and (16), then for v^a_α, w^a and p^a from Eq. (56), and obtain the following system of homogeneous partial differential equations in V^a_α, W^a and P^a.

$$\begin{aligned} &\mu(V^a_{\alpha,\beta\beta} + V^a_{\beta,\alpha\beta}) - P^a_{,\alpha} - D_{ab}\mu(D_{cb}V^c_\alpha + W^b_{,\alpha}) = -\rho\omega^2 V^a_\alpha, \\ &\mu(D_{ca}V^c_{\alpha,\alpha} + W^a_{,\alpha\alpha}) + D_{ab}P^b - 2D_{ab}\mu D_{cb}W^c = -\rho\omega^2 W^a, \\ &V^a_{\alpha,\alpha} + D_{ca}W^c = 0. \end{aligned} \tag{57}$$

Eqs. (57)define an eigenvalue problem for the determination of the frequency ω and the corresponding mode vector V^a_α, W^a and P^a. These will depend upon boundary conditions imposed at the plate edges. No initial conditions are needed for a free vibration problem. For $a = 0, 1, 2, \ldots, K$, Eqs. (57) under the appropriate boundary conditions can be solved numerically either by the finite element method, e.g., see [23] or by a meshless method [18–20] where plates made of compressible linear elastic materials have been studied. Since the plate theory proposed herein takes pressures and displacements as unknowns, the solution of the problem by a numerical method should not exhibit the locking phenomenon.

For $K=1$, Eqs. (57) give

$$V^0_{\alpha,\alpha}+\sqrt{3}W^1=0,$$
$$\mu(V^0_{\alpha,\beta\beta}+V^0_{\beta,\alpha\beta})-P^0_{,\alpha}=-\rho\omega^2V^0_\alpha,$$
$$\mu W^0_{,\alpha\alpha}=-\rho\omega^2W^0; \tag{58}$$

$$V^1_{\alpha,\alpha}=0,$$
$$\mu V^1_{\alpha,\beta\beta}-P^1_{,\alpha}-\sqrt{3}\mu W^0_{,\alpha}-3\mu V^1_\alpha=-\rho\omega^2V^1_\alpha,$$
$$\mu W^1_{,\alpha\alpha}+\sqrt{3}P^0-6\mu W^1=-\rho\omega^2W^1. \tag{59}$$

For $K>1$, one can similarly deduce equations analogous to Eqs. (58) and (59).

5.1. Simply supported rectangular plate

For a simply supported rectangular plate whose mid-surface occupies the region $[0,\ L_1]\times[0,\ L_2]$, the boundary conditions (38) can be written as

$$W^a=0,\quad M^a_{11}=0,\ \ M^a_{21}=0\quad \text{on } x_1=0,\ L_1,$$

$$W^a=0,\quad M^0_{22}=0,\ \ M^a_{12}=0\quad \text{on } x_2=0,\ L_2. \tag{60}$$

These are identically satisfied by the following choice for V^a_α, W^a and P^a.

$$V^a_1=\sum_{m,n=0}^{\infty}\tilde{V}_1^{amn}\cos\frac{m\pi x_1}{L_1}\sin\frac{n\pi x_2}{L_2},$$
$$V^a_2=\sum_{m,n=0}^{\infty}\tilde{V}_2^{amn}\sin\frac{m\pi x_1}{L_1}\cos\frac{n\pi x_2}{L_2},$$
$$W^a=\sum_{m,n=0}^{\infty}\tilde{W}^{amn}\sin\frac{m\pi x_1}{L_1}\sin\frac{n\pi x_2}{L_2},$$
$$P^a=\sum_{m,n=0}^{\infty}\tilde{P}^{amn}\sin\frac{m\pi x_1}{L_1}\sin\frac{n\pi x_2}{L_2}, \tag{61}$$

where m and n are integers. As pointed out in [22] the lower limit for m and n in Eq. (61) should be zero and not one as has been assumed in several works. For the zeroth order plate theory, Eqs. (61) and (58) with $W^1=0$ give

$$\omega^2=\frac{\mu}{\rho}\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right),$$

$$\frac{m}{L_1}\tilde{V}_1^{0mn}+\frac{n}{L_2}\tilde{V}_2^{0mn}=0,\quad \tilde{P}^{0mn}=0. \tag{62}$$

That is, during free vibrations of a simply supported rectangular plate the zeroth-order plate theory does not delineate the effect of material incompressibility, and computed frequencies are the same as those for a plate made of a compressible material. Frequencies given by Eq. (62) agree with those derived from Eq. (13) of Ref. [1] with $\alpha=0$, and the corresponding mode shape may or may not approximate the analytical one since Eqs. $(58)_3$ and $(61)_3$ do not require that W^0 identically vanish as is the case for the analytical solution. We should add that for a clamped rectangular plate, the zeroth-order plate theory may give different frequencies for a plate made of a compressible and an incompressible material.

For $K=1$, substitution from Eq. (61) into Eqs. (58) and (59) gives the following eigenvalue problem for the determination of the frequency ω and the eigenvector (or the mode shape) V^0_α, V^1_α, W^0, W^1, P^0 and P^1.

$$\frac{m}{L_1}\tilde{V}_1^{0mn}+\frac{n}{L_2}\tilde{V}_2^{0mn}-\sqrt{3}\tilde{W}^{1mn}=0,$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{V}_1^{0mn}+\frac{m\pi}{L_1}\tilde{P}^{0mn}=\rho\omega^2\tilde{V}_1^{0mn},$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{V}_2^{0mn}+\frac{n\pi}{L_2}\tilde{P}^{0mn}=\rho\omega^2\tilde{V}_2^{0mn},$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{W}^{0mn}=\rho\omega^2\tilde{W}^{0mn},$$
$$\left(\frac{m}{L_1}\tilde{V}_1^{1mn}+\frac{n}{L_2}\tilde{V}_2^{1mn}\right)=0,$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{V}_1^{1mn}+\sqrt{3}\mu\left(\frac{m\pi}{L_1}\right)\tilde{W}^{0mn}+3\mu\tilde{V}_1^{1mn}+\frac{m\pi}{L_1}\tilde{P}^{1mn}=\rho\omega^2\tilde{V}_1^{1mn},$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{V}_2^{1mn}+\sqrt{3}\mu\left(\frac{n\pi}{L_2}\pi^2\right)\tilde{W}^{0mn}+3\mu\tilde{V}_2^{1mn}+\frac{n\pi}{L_2}\tilde{P}^{1mn}=\rho\omega^2\tilde{V}_2^{1mn},$$
$$\mu\pi^2\left(\frac{m^2}{L_1^2}+\frac{n^2}{L_2^2}\right)\tilde{W}^{1mn}+6\mu\tilde{W}^{1mn}-\sqrt{3}\tilde{P}^{0mn}=\rho\omega^2\tilde{W}^{1mn}. \tag{63}$$

For different integer values of m and n, one can solve the eigenvalue problem (63) and find the spectrum of frequencies and the corresponding eigenvectors.

For a given value of K one gets $4(K+1)$ algebraic equations analogous to Eqs. (63) for the $4(K+1)$ unknowns. For assigned values of m and n, these algebraic equations can be solved for the frequencies and the corresponding mode shapes. The following question naturally arises: How does the solution accuracy and the corresponding computational effort compare with the corresponding quantities in the analysis of the 3-D problem by the finite element method (FEM). For a plate made of a compressible linear elastic material Batra and Aimmanee [23] found that the seventh-order plate theory requires less computational resources than the FE analysis of the 3-D problem, and gives equally good results for frequencies of flexural modes of vibration and improved results for the in-plane modes of vibration. Recall that in the Kth-order plate theory, displacements and the pressure are taken to be Kth-order polynomials in z but in the FEM using 8-node brick elements they will be approximated by piecewise linear polynomials. For K linear elements through the plate thickness the number of unknowns and the linear algebraic equations in the analyses of the problem by the FEM and the plate theory equations will be nearly the same. However, K piecewise polynomials of degree 1 in z will not approximate well the Kth-order polynomial in z. Recall that in-plane motions of points of the midsurface of the plate are expressed in polynomials of the same degree both in the FEM and the plate-theory equations. Note that the CPU time required to invert $(K+1)$ $n \times n$ matrices is significantly less than that needed to invert one $(K+1)n \times (K+1)n$ matrix.

Batra and Aimmanee [24] have used a higher order plate theory and Poisson's ratio equal to 0.49 to compute frequencies of incompressible rectangular plates under different types of boundary conditions.

6. Remarks

It is very likely that plates made of rubberlike materials will experience large strains. Thus a plate theory for these materials should incorporate both geometric and material nonlinearities. A possibility is to divide the load into several increments, and for each incremental load develop a theory of infinitesimal deformations of pre-stressed shells since an initially flat plate will be deformed into a part of a shell by the first incremental load. Alternatively, one can use a total Lagrangian description of motion in which case equations for incremental deformations are always referred to the initial configuration of the plate. Plate theory equations will be considerably more involved than those for infinitesimal deformations.

For a thin plate, i.e., $R \ll 1$, one usually assumes that the transverse normal stress, σ_{33}, is negligible as compared to σ_{11} and/or σ_{22}. For an incompressible material $\sigma_{33} = 0$ implies that

$$p^a = C_{33kl} e^a_{kl}, \tag{64}$$

which can be solved for e^a_{33}. Consistent with the classical thin plate theories one can set $w^1 = w^2 = \cdots = w^k = 0$ in Eq. $(11)_2$, and substitute for e^a_{33} from Eq. (64) into the constitutive relation (6). Eqs. (23)–(25) with $T^a_3 = 0$ govern deformations of the plate.

Assuming that $C_{3333} \neq 0$, Eq. (64) can be solved for e^a_{33} and the result substituted in Eq. (5) to obtain

$$\sigma^a_{ij} = -\tilde{p}^a + \tilde{C}_{ijkl}(e^a_{kl} - e^a_{33}\delta_{k3}\delta_{l3}). \tag{65}$$

where $\tilde{C}_{ijkl}$ are the modified elasticities. There is no need to introduce the strain terms in the pressure field because like p^a, $\tilde{p}^a$ cannot be determined from the strain field.

When using the elasticity theory one solves $\sigma_{33} = 0$ for the pressure field and substitutes for it in the governing equations to find the deformation field. One could do so here and substitute for p^a from Eq. (64) into Eqs. (27) and (28). However, in this case one must consider $w^1, w^2, \ldots, w^k$ in Eq. $(11)_2$.

As has been demonstrated in [2,5,18,20,23] transverse stresses computed from this plate theory equations agree well with those obtained from the analytical solution of the problem. Furthermore there is no need to introduce the shear correction factor commonly used in the Reissner-Mindlin (or the first-order shear deformation) plate theories.

We note that lengths have been non-dimensionalized with respect to $h/2$ where h equals the plate thickness. That is why h does not appear in Eq. $(62)_1$ and some other equations.

7. Conclusions

We have used the principle of virtual work to derive a compatible higher-order shear and normal deformable theory for a plate made of an incompressible linear elastic material. The difference between compressible and incompressible materials is that only volume preserving deformations are admissible in incompressible materials, and one needs to find the hydrostatic pressure as a part of the solution of the pertinent initial boundary value problem. This is reflected in the plate theory by also expanding the hydrostatic pressure as a power series in the thickness coordinate. Governing equations for plate theories of different order are deduced and free vibrations of plates are studied. It is found that frequencies of a simply supported rectangular plate computed from the zeroth-order plate theory agree with those found from the analytical solution but the corresponding two mode shapes need not match with each other.

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References

[1] Aimmanee S, Batra RC. Analytical solution for vibration of an incompressible isotropic linear elastic plate, and frequencies missed in previous solutions. J Sound Vib 2007;302:613–20.
[2] Batra RC, Vidoli S. Higher order piezoelectric plate theory derived from a three-dimensional variational principle. AIAA J 2002;40(1): 91–104.
[3] Mindlin RC, Medick MA. Extensional vibrations of thick plates. J Appl Mech 1959;26:145–51.
[4] Yang JS, Batra RC. Mixed variational principles in nonlinear piezoelectricity. Int J Nonlinear Mech 1995;30:719–26.
[5] Batra RC, Vidoli S, Vestroni F. Plane waves and modal analysis in higher-order shear and normal deformable plate theories. J Sound Vib 2002;257(1):63–88.
[6] Herrmann LR. Elasticity equations for nearly incompressible materials by a variational theorem. AIAA J 1965;3:1896–900.
[7] Vlasov VZ. The method of initial function in problems of theory of thick plates and shells. In: Proceedings of the 9th international Congress of applied mechanics, vol. 6. Brussels: Universite de Bruxelles; 1957. p. 321–30.
[8] Lo KH, Christensen RM, Wu EM. A higher-order theory of plate deformation, part I: homogeneous plates. J Appl Mech 1977;44(4): 669–76.
[9] Kant T. Numerical analysis of thick plates. Comput Methods Appl Mech Eng 1982;31(1):1–18.

[10] Hanna NF, Leissa AW. A higher order shear deformation theory for the vibration of thick plates. J Sound Vib 1994;170(4):545–55.
[11] Reddy JN. A simple higher-order theory for laminated composite plates. J Appl Mech 1984;51(4):745–52.
[12] Lee PC-Y, Yu JD. Governing equations of piezoelectric plates with graded properties across the thickness. IEEE Trans Ultrason Ferroelectrics Freq Control 1998;45(1):236–50.
[13] Lee PC-Y, Syngellakis S, Hou JP. A two dimensional theory for high-frequency vibrations of piezoelectric crystal plates with or without electrodes. J Appl Mech 1987;61(4):1249–62.
[14] Batra RC. Elements of continuum mechanics. Reston, VA: AIAA; 2005.
[15] Vel SS, Batra RC. Exact solutions for thermoelastic deformations of functionally graded thick rectangular plates. AIAA J 2002;40(7):1421–33.
[16] Vel SS, Batra RC. Three-dimensional exact solution for the vibration of functionally graded rectangular plates. J Sound Vib 2004;272:703–30.
[17] Vel SS, Batra RC. Three-dimensional analysis of transient thermal stresses in functionally graded plates. Int J of Solids Struct 2003;40: 7181–96.
[18] Qian LF, Batra RC. Transient thermoelastic deformations of a thick functionally graded plate. J Thermal Stresses 2004;27:705–40.
[19] Gilhooley DF, Batra RC, Xiao JR, McCarthy MA, Gillespie Jr JW, Analysis of thick functionally graded plates by using higher-order shear and normal deformable plate theory and MLPG method with radial basis functions. Compos Struc 2007;80:539–52.
[20] Qian LF, Batra RC. Design of bidirectional functionally graded plate for optimal natural frequencies. J Sound Vib 2005;280:415–24.
[21] Batra RC, Jin J. Natural frequencies of a functionally graded anisotropic rectangular plate. J Sound Vib 2005;282:509–16.
[22] Batra RC, Aimmanees. Missing frequencies in previous exact solutions of free vibrations of simply supported rectangular plates. J Sound Vib 2003;265:887–96.
[23] Batra RC, Aimmanee S. Vibrations of thick isotropic plates with higher order shear and normal deformable plate theories. Comput Struc 2005;83:934–55.
[24] Batra RC, Aimmanee S. Vibration analysis of incompressible isotropic rectangular plates with a higher order shear and normal deformable theory. J Sound Vib 2007; 307:961–71.