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Short Communication

Vibration of an incompressible isotropic linear elastic rectangular plate with a higher-order shear and normal deformable theory

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Abstract

We use a mixed higher-order shear and normal deformable plate theory (HOSNDPT) of Batra and Vidoli with Poisson's ratio equal to 0.49 and the finite element method to analyze vibrations of a homogeneous isotropic rectangular plate made of an incompressible linear elastic material. Through-the-thickness integrals are evaluated exactly, and those over an element in the midplane of the plate are evaluated by using the 2×2 Gauss quadrature rule. The plate theory equations are used to ascertain frequencies of a clamped–clamped and a clamped–free square/rectangular plate of different aspect ratios. Through-the-thickness modes of vibration valid for compressible and incompressible materials and missed by previous investigators are also identified. Computed frequencies are found to match well with those deduced from the analytical solution.

1. Introduction

Because of the increasing use of rubberlike materials in aerospace and automotive industries, and the realization that many biological materials can be modeled as incompressible, we study free vibrations of a plate made of an isotropic incompressible linear elastic material. Of course, only isochoric (i.e. volume preserving) deformations are admissible in an incompressible body; thus it can undergo only pure distortional deformations. Corresponding to the constraint of incompressibility, the constitutive relation involves a hydrostatic pressure that cannot be determined from the deformation field but is found from a solution of the balance of linear momentum and the normal traction boundary conditions prescribed at least on a part of the boundary of the body. For free vibrations of a plate, the top and the bottom surfaces are traction free, and there are no body forces. The analytical solution for a simply supported rectangular plate given in Ref. [1] provides benchmark frequencies for comparison with those computed from a plate theory.

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Here, we employ the mixed higher-order shear and normal deformable plate theory (HOSNDPT) of Batra and Vidoli [2], set Poisson's ratio = 0.49, and compare computed frequencies for a simply supported square and a rectangular plate of aspect ratios (thickness/larger in-plane dimension) 1/4, 1/8, 1/12, and 1/20 with those obtained analytically. Note that Poisson's ratio = 0.5 for an incompressible material. The plate theory satisfies exactly the boundary condition of null surface tractions on the top and the bottom surfaces of a free plate, and incorporates constitutive relations of three-dimensional linear elasticity. For a Kth order plate theory, the three components of displacement and the three in-plane stresses are expanded in the thickness (z) direction upto terms of order z^K but the transverse shear and the transverse normal stresses have terms of the order z^{K+2}. Batra and Aimmanee [3] used this theory in conjunction with the finite element method to analyze frequencies of a rectangular plate made of a compressible material. We show here that with through-the-thickness integrals evaluated exactly, integrals over a 4-node quadrilateral element on the midsurface by a 2×2 Gauss integration rule, and Poisson's ratio = 0.49, the HOSNDPT gives frequencies in close agreement with those found from the analytical solution for a plate made of an incompressible material. The plate theory equations are used to find natural frequencies of a clamped–clamped and a clamped–free rectangular plate made of an incompressible material for which analytical solutions are difficult to find.

2. Formulation of the problem

In rectangular Cartesian coordinates and in the absence of body forces infinitesimal deformations of an incompressible body are governed by the following balance of mass and the balance of linear momentum:

$$u_{i,i} = 0, \tag{1}$$

$$\rho \ddot{u}_i = \sigma_{ij,j}, \quad i,j = 1,2,3. \tag{2}$$

Here $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{u}$ the displacement, $\rho > 0$ the mass density, a superimposed dot indicates differentiation with respect to time t, $\sigma_{ij,j} = \partial \sigma_{ij}/\partial x_j$, $\mathbf{x}$ the present position of a material point, and a repeated index implies summation over the range of the index. Eq. (1) implies that deformations are isochoric or volume preserving and hence the mass density stays constant. The balance of moment of momentum is identically satisfied by requiring that the stress tensor $\boldsymbol{\sigma}$ is symmetric. For an incompressible linear elastic isotropic material:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \tag{3}$$

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \tag{4}$$

where p is the hydrostatic pressure not determined from the infinitesimal strain tensor $\mathbf{e}$, δ_{ij} is the Kronecker delta, and $\mu > 0$ is the shear modulus. Substitution for $\mathbf{e}$ from Eq. (4) into Eq. (3) and for $\boldsymbol{\sigma}$ from Eq. (3) into Eq. (2) gives:

$$\rho \ddot{u}_i = -p_{,i} + \mu u_{i,jj}. \tag{5}$$

Here we have assumed that the body is homogeneous; thus μ and ρ are constants.

For a simply supported rectangular plate occupying the region $[0, L_x] \times [0, L_y] \times [0, h]$, boundary conditions are listed below:

$$\begin{aligned} &u_2 = u_3 = 0, \quad \sigma_{11} = 0 \quad \text{on} \quad x_1 = 0,\ L_x, \\ &u_1 = u_3 = 0, \quad \sigma_{22} = 0 \quad \text{on} \quad x_2 = 0,\ L_y, \\ &\sigma_{i3} = 0 \quad \text{on} \quad x_3 = 0,\ h. \end{aligned} \tag{6}$$

Thus the top and the bottom surfaces of the plate are traction free. The lateral deflection u_3 and the normal tractions vanish on all four edge surfaces. Boundary conditions in Eq. (6) are not easily realized in a laboratory where the plate edges are typically supported on rollers or sharp-knife wedges. However, they have been widely used since Srinivas et al. [4] presented analytical solutions for free vibrations of a rectangular plate made of a compressible linear elastic material. At a free edge the three components of the traction vector vanish, and at a clamped edge the three components of the displacement vector equal zero.

For the steady-state vibration problem no initial conditions are needed.

3. Brief review of the mixed higher-order shear and normal deformable plate theory (HOSNDPT)

In this section Greek indices range over 1 and 2, Latin indices over 1, 2 and 3, and a repeated index implies summation over the range of the index. Recall that for studying free vibrations of a plate, the body force and surface tractions identically vanish. We decompose as follows the position vector **x** of a point, the displacement **u**, and the outward unit normal **n**:

$$x_i = x_\alpha \delta_{i\alpha} + z\delta_{i3}, \quad u_i = v_\alpha \delta_{i\alpha} + w\delta_{i3}, \quad n_i = \hat{n}_\alpha \delta_{i\alpha} + n\delta_{i3}. \tag{7}$$

Here δ_{ij} is the Kronecker delta, and $z = x_3$. The strain tensor for infinitesimal strains is given by

$$e_{ij} = \hat{e}_{\alpha\beta}\delta_{i\alpha}\delta_{j\beta} + \gamma_\alpha(\delta_{i\alpha}\delta_{j3} + \delta_{i3}\delta_{j\alpha}) + \varepsilon\delta_{i3}\delta_{j3}, \tag{8}$$

where

$$\begin{aligned} &\hat{e}_{\alpha\beta} = (v_{\alpha,\beta} + v_{\beta,\alpha})/2, \quad e_{\alpha 3} \equiv \gamma_\alpha = (v'_\alpha + w_{,\alpha})/2, \quad e_{33} \equiv \varepsilon = w', \\ &w' = \partial w/\partial z = \partial w/\partial x_3, \quad v'_\alpha = \partial v_\alpha/\partial z. \end{aligned} \tag{9}$$

Thus γ_α and ε denote, respectively, the transverse shear strains and the transverse normal strain. Analogous to the decomposition in Eq. (8) of the infinitesimal strain tensor, we write:

$$\sigma_{ij} = \hat{\sigma}_{\alpha\beta}\delta_{i\alpha}\delta_{j\beta} + \sigma^t_\alpha(\delta_{i\alpha}\delta_{j3} + \delta_{i3}\delta_{j\alpha}) + \sigma^n\delta_{i3}\delta_{j3}, \tag{10}$$

where $\hat{\sigma}_{\alpha\beta}$, σ^t_α and σ^n are, respectively, the in-plane components of the stress tensor, the transverse shear stresses and the transverse normal stress.

For a compressible isotropic linear elastic material

$$\begin{aligned} &\begin{Bmatrix} \hat{e}_{11} \\ \hat{e}_{22} \\ 2\hat{e}_{12} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -v & 0 \\ -v & 1 & 0 \\ 0 & 0 & 2(1+v) \end{bmatrix}\begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{12} \end{Bmatrix} + \frac{\sigma^n}{E}\begin{Bmatrix} -v \\ -v \\ 0 \end{Bmatrix}, \\ &\begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} = \frac{2(1+v)}{E}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \end{Bmatrix}, \\ &\varepsilon = -\frac{v}{E}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{12} \end{Bmatrix} + \frac{\sigma^n}{E}, \end{aligned} \tag{11}$$

where E is Young's modulus, v Poisson's ratio and $2\mu = E/(1+v)$. The constitutive relation for an incompressible material is obtained from Eq. (11) by setting $v = 1/2$. For $v = 1/2$, Eq. (11) gives $\hat{e}_{11} + \hat{e}_{22} + \varepsilon = 0$ and consequently there is no volume change; however Eq. (11) cannot be inverted to solve for stresses in terms of strains. One usually takes $v = 0.49$ or a similar value close to 0.5 and computes results for an incompressible material.

With the transformation

$$\xi = (2z - h)/h, \tag{12}$$

we use orthonormal Legendre polynomials $L_0(\xi), L_1(\xi), \ldots, L_K(\xi)$ defined on $[-1, 1]$ and satisfying

$$\int_{-1}^{1} L_a(\xi)L_b(\xi)\,\mathrm{d}\xi = \delta_{\mathrm{ab}}, \quad a, b = 0, 1, 2, \ldots, K \tag{13}$$

as basis functions to expand displacements and stresses in powers of z. Unless stated otherwise, indices a and b range over $0, 1, 2, \ldots, K$, and a repeated index is summed irrespective of its appearance as a subscript or a superscript or it being enclosed in parentheses. Note that

$$L'_a(\xi) = \frac{\mathrm{d}L_a}{\mathrm{d}z}(\xi) = \frac{\mathrm{d}L_a}{\mathrm{d}\xi}\frac{2}{h} = \frac{2}{h}\sum_{b=0}^{(a-1)} D_{\mathrm{ab}}L_b, \tag{14}$$

where D_{ab} are constants. We set

$$\begin{aligned} v_\alpha(x_\beta, z, t) &= L_a(\xi) v_\alpha^{(a)}(x_\beta, t), \\ w(x_\beta, z, t) &= L_a(\xi) w^{(a)}(x_\beta, t), \\ \hat{\sigma}_{\alpha\beta}(x_\gamma, z, t) &= L_a(\xi) N_{\alpha\beta}^{(a)}(x_\gamma, t), \\ \sigma_\alpha^t(x_\gamma, z, t) &= \tilde{L}_a(\xi) T_\alpha^{(a)}(x_\gamma, t), \\ \sigma^n(x_\gamma, z, t) &= \tilde{L}_a(\xi) \Sigma^{(a)}(x_\gamma, t), \end{aligned} \tag{15}$$

where

$$\int_{-1}^{1} \tilde{L}_a(\xi) L_b(\xi)\, \mathrm{d}\xi = \delta_{\mathrm{ab}}, \tilde{L}_a(\pm 1) = 0. \tag{16}$$

In Eqs. $(15)_4$ and $(15)_5$, $\tilde{L}_a(\xi)$, $a = 0, 1, 2, \ldots, K$ are modified Legendre polynomials satisfying Eqs. $(16)_1$ and $(16)_2$, $\tilde{L}_a(\xi)$ is a polynomial of degree ξ^{a+2}.

It is evident from Eqs. $(15)_1$–$(15)_3$ that all three components of displacement and in-plane stresses are expanded in z upto the power z^K since $1, z, z^2, \ldots, z^K$ are basis functions equivalent to $L_0(\xi), L_1(\xi), \ldots, L_K(\xi)$. However, the transverse normal and the transverse shear stresses have terms of the order z^{K+2}. Also, because of the requirement in Eq. $(16)_2$, Eqs. $(15)_4$, and $(15)_5$ satisfy exactly the condition of vanishing surface tractions on the top and the bottom surfaces of the plate. When using a mixed variational principle such as the Hellinger–Reissner principle to derive the plate equations, one can presume independent expansions for stresses and displacements.

Expressions for the Legendre polynomials $L_a(\xi)$ and for the modified Legendre polynomials $\tilde{L}_a(\xi)$ are given in Ref. [5]. Eqs. $(15)_4$ and $(15)_5$ need to be modified when either the top or the bottom or both of these surfaces have non-zero surface tractions acting on them.

We refer the reader to Refs. [2,5,6] for details of deriving the balance laws, constitutive relations, and initial and boundary conditions for the two-dimensional Kth-order plate theory, and simply list them below.

Equations of motion:

$$\begin{aligned} &N_{\alpha\beta,\beta}^{(a)} - D_{\mathrm{ab}} T_\alpha^{(b)} = R_{\mathrm{ab}} \ddot{v}_\alpha^{(b)} \quad \text{on } S,\ a = 0, 1, 2, \ldots, K;\ \alpha = 1, 2; \\ &T_{\alpha,\alpha}^{(a)} - D_{\mathrm{ab}} \Sigma^{(b)} = R_{\mathrm{ab}} \ddot{w}^{(b)} \quad \text{on } S. \end{aligned} \tag{17}$$

Constitutive relations:

$$\begin{aligned} \{\hat{e}^{(a)}\} &= [C^{\mathrm{pp}}]\{N^{(a)}\} + [C^{\mathrm{pt}}]\{T^{(a)}\} + \{C^{\mathrm{pn}}\}\Sigma^{(a)}, \\ \{\gamma^{(a)}\} &= [C^{\mathrm{tp}}]\{N^{(a)}\} + [C^{\mathrm{tt}}]P_{\mathrm{ab}}\{T^{(b)}\} + \{C^{\mathrm{tn}}\}P_{\mathrm{ab}}\Sigma^{(b)}, \\ \varepsilon^{(a)} &= [C^{\mathrm{np}}]\{N^{(a)}\} + [C^{\mathrm{nt}}]P_{\mathrm{ab}}\{T^{(b)}\} + C^{\mathrm{nn}}P_{\mathrm{ab}}\Sigma^{(b)}. \end{aligned} \tag{18}$$

Boundary conditions:

$$\begin{aligned} &N_{\alpha\beta}^{(a)} \hat{n}_\beta = F_\alpha^{(a)}, \quad T_\alpha^{(a)} \hat{n}_\alpha = F_3^{(a)} \quad \text{on } \partial_t S, \\ &v_\alpha^{(a)} = \bar{v}_\alpha^{(a)}, \quad w^{(a)} = \bar{w}^{(a)} \quad \text{on } \delta_u S. \end{aligned} \tag{19}$$

Initial conditions:

$$\begin{aligned} v_\alpha^{(a)}(x_\beta, 0) &= \overset{\circ}{v}{}_\alpha^{(a)}(x_\beta), \\ w_\alpha^{(a)}(x_\beta, 0) &= \overset{\circ}{w}{}_\alpha^{(a)}(x_\beta). \end{aligned} \tag{20}$$

Here,

$$R_{\mathrm{ab}} = \frac{h}{2}\int_{-1}^{1} \rho L_a L_b\, \mathrm{d}\xi = \rho \frac{h}{2}\delta_{\mathrm{ab}}, \quad P_{\mathrm{ab}} = \frac{h}{2}\int_{-1}^{1} \tilde{L}_a \tilde{L}_b\, \mathrm{d}\xi, \tag{21}$$

S is the midsurface of the plate, $\partial_t S$ and $\partial_u S$ are parts of the boundary of S where surface tractions and displacements are prescribed, respectively. $N^{(0)}_{\alpha\beta}$ is the membranal stress tensor, $N^{(1)}_{\alpha\beta}$ the matrix of bending moments, and matrices $N^{(a)}_{\alpha\beta}$ ($a = 1, 2, \ldots, K$) are a linear combination of matrices of bending moments of order zero through a. $T^{(0)}_{\alpha}$ is the resultant transverse shear force, $T^{(1)}_{\alpha}$ the moment of internal double forces acting along the x_3-axis, and $T^{(a)}_{\alpha}$ equals the linear combination of moments up to the ath order of internal double forces. $\Sigma^{(0)}$ is the transverse normal force, and $\Sigma^{(a)}$ the linear combination of moments up to the ath order of the transverse normal force. Neither the transverse normal strain nor the transverse normal stress are assumed to vanish. Constitutive and kinematic relations of the three-dimensional linear elasticity theory are used. The presence of P_{ab} in Eqs. $(18)_2$ and $(18)_3$ implies that $\gamma^{(a)}$ and $\varepsilon^{(a)}$ depend upon $\mathbf{T}^{(b)}$ and $\Sigma^{(b)}$ for $0 \leqslant b \leqslant K$; thus equations for transverse strains and moments of transverse forces are strongly coupled. Because of the presence of $\mathbf{T}^{(a)}$ in Eqs. $(17)_1$ and $(17)_2$ they are coupled. Also the occurrence of D_{ab} in these two equations suggests that equations for $a = K$ involve $\mathbf{T}^{(0)}, \mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(K-1)}$.

Substitution for $\mathbf{v}$ and w from Eqs. $(15)_1$ and $(15)_2$ into Eq. (9) gives expressions for the infinitesimal strains. Substituting these and the expressions from Eqs. $(15)_3$–$(15)_5$ for stresses into the constitutive relation in Eq. (11) and comparing the result with Eqs. (18) we can derive expressions for the matrices C^{pp}, C^{pt} etc. of elastic constants. For $v \neq 1/2$ and $a = 0, 1, 2, \ldots, K$ Eqs. $(18)_1$–$(18)_3$ can be solved for $\mathbf{N}^{(a)}, \mathbf{T}^{(a)}$ and Σ^a in terms of strains, E and v; the results are given in Appendix C of Ref. [3]. When these expressions are substituted in Eqs. $(17)_1$ and $(17)_2$ we get a set of second-order partial differential equations for $\mathbf{v}^{(a)}$ and $w^{(a)}$ which can be solved under boundary conditions (19) and initial conditions (20).

For free vibrations of a plate, initial conditions (20) are not needed and boundary conditions (19) are such that

$$F^{(a)}_{\alpha}\bar{v}^{(a)}_{\alpha} = 0, \quad F^{(a)}_{3}\bar{w}^{(a)} = 0; \text{ no sum on } a. \tag{22}$$

That is, no work is done by external forces prescribed on the boundary ∂S of the midsurface S of the plate. For simply supported (SP), clamped (C) and free (F) edges of the plate, boundary conditions are listed below:

$$\begin{aligned} &\text{SP} : N^{(a)}_{11} = 0, \quad w^{(a)} = 0, \quad v^{(a)}_2 = 0 \quad \text{on } x_1 = 0, L_x,\\ &N^{(a)}_{22} = 0, \quad w^{(a)} = 0, \quad v^{(a)}_1 = 0 \quad \text{on } x_2 = 0, L_y; \end{aligned} \tag{23}$$

$$\text{C} : v^{(a)}_1 = v^{(a)}_2 = w^{(a)} = 0, \quad \text{on } x_1 = 0,\ L_x, \quad x_2 = 0,\ L_y; \tag{24}$$

$$\begin{aligned} &\text{F} : N^{(a)}_{11} = 0, \quad N^{(a)}_{21} = 0, \quad T^{(a)}_1 = 0 \quad \text{on } x_1 = 0, \quad L_x,\\ &N^{(a)}_{12} = 0, \quad N^{(a)}_{22} = 0, \quad T^{(a)}_2 = 0 \quad \text{on } x_2 = 0,\ L_y. \end{aligned} \tag{25}$$

Also for free vibrations we seek solutions of Eqs. (17), (18), and (23) or (24) or (25) of the form:

$$\begin{aligned} v^{(a)}_{\alpha}(x_\beta, t) &= V^{(a)}_{\alpha}(x_\beta)\mathrm{e}^{\mathrm{i}\omega t},\\ w^{(a)}(x_\beta, t) &= W^{(a)}(x_\beta)\mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{26}$$

where ω is a natural frequency. For $a = 0, 1, 2, \ldots, K$ we get an eigenvalue problem which is solved by the finite element method. Details of deriving a weak formulation of the problem and solving the resulting eigenvalue problem are given in Ref. [3].

4. Results

Numerical results have been computed by dividing the midsurface S of the plate into a finite element mesh comprised of 4-node isoparametric quadrilateral elements, consistent mass matrix, no shear-connection factor and 2×2 integration rule to evaluate various integrals appearing in the weak formulation of the problem. The incompressibility of the plate material has been approximated by setting $v = 0.49$. For higher values of v, e.g., $v = 0.499$ or $v = 0.4999$, the computer code failed to give stable results.

We assume that the plate is made of a rubberlike material. Material properties of rubber listed below are taken from the website www.efunda.com.

$$E = 1\,\text{MPa}, \quad \rho = 1000\,\text{kg/m}^3. \tag{27}$$

The non-dimensional natural frequencies Ω_{mn}, are defined by

$$\Omega_{mn} = \omega_{mn} h \sqrt{\frac{\rho}{\mu}}, \tag{28}$$

where subscripts m and n denote half-wave numbers in x- and y-directions, respectively. The minimum order of the plate theory needed to find all frequencies depends upon the aspect ratio of the plate, and the boundary conditions at the edges.

4.1. Simply supported square plate

For a simply supported square plate and different values of h/L_x, we have compared in Table 1 the first ten natural frequencies of a simply supported square plate with their analytical values. The notations "o" and "i" represent, respectively, the out-of-plane modes with $u_3(x_1, x_2, x_3 = h/2) \neq 0$ for some x_1 and x_2, and the in-plane modes with $u_3(x_1, x_2, x_3 = h/2) = 0$ for every x_1 and x_2. Batra and Aimmanee [7] have pointed out that the in-plane modes of vibration were missed by several investigators; e.g. see Refs. [8–11]. The plate is usually called thin when $h/L_x < 0.1$. The order, K, of the plate theory equals 5 for $h/L_x = 1/4$ and 1/8, and 3 for $h/L_x = 1/12$ and 1/20. The number of out-of-plane modes increases with a decrease in the value of h/L_x. A frequency computed with the plate theory agrees very well with the corresponding analytical value except when either m or n is large. This difference can be reduced by refining the finite element mesh.

4.2. Simply supported rectangular plates made of compressible/incompressible materials

We now compare frequencies of geometrically identical plates made of compressible and incompressible materials. We have listed in Table 2 frequencies of a rectangular plate and $L_x = 2L_y$ with Poisson's ratio equal to either 0.3 or 0.49. Results for $v = 0.3$ are taken from Ref. [3] where in-plane and out-of-plane modes of vibration were indicated by letters a and s, respectively, and were erroneously called antisymmetric and symmetric. Whereas only pure distortional modes of vibration are admissible in a plate made of an incompressible material, there is no restriction on the modes of vibration that may occur in a plate made of a compressible material. Both for compressible and incompressible materials the plate theory gives first ten frequencies that are close to those obtained from the analytical solution of the problem. For $h/L_x = 1/4$ the first ten mode shapes for plates made of compressible and incompressible materials coincide with each other except that their order is different. For example, $(m, n) = (0, 1)$ gives 3rd and 4th lowest frequency for a plate made of an incompressible material but 4th one for a plate comprised of a compressible material. For $h/L_x = 1/8$ the 10th frequency corresponds to $(m, n) = (3, 0)$ for an incompressible material but to $(m, n) = (4, 1)$ for a compressible material. Also for an incompressible material there are two frequencies among the first ten modes of vibration for $(m, n) = (2, 1)$—one corresponding to in-plane mode of deformation and the other to the out-of-plane mode of deformation. However, for a plate made of a compressible material, only the frequency corresponding to the out-of-plane mode of vibration is included in the first ten frequencies.

4.3. Clamped square and rectangular plates

The first ten natural frequencies of clamped square and rectangular plates made of incompressible materials are listed in Table 3. As for a simply supported plate, the number of in-plane modes of vibration corresponding to the ten lowest natural frequencies decreases with a decrease in the value of h/L_x. As expected the frequency of a clamped plate is higher than that of the corresponding simply supported plate. For a given value of h/L_x the simply supported plate admits more in-plane modes of vibration than a clamped plate corresponding to the first ten lowest frequencies. For example, for a square plate with $h/L_x = 1/8$ there are five in-plane modes of vibration for a simply supported plate and only two for a clamped plate. Recall that the midsurface of the plate has null deflections in an in-plane mode of vibration.

Table 1
Comparison of the first 10 normalized natural frequencies of a simply supported square plate made of a linear elastic isotropic incompressible material computed from the analytical solution with those obtained from the mixed higher-order shear and normal deformable plate theory

Number	Mode		Normalized frequencies		
	(m,n)	in-plane/out-of-plane	Analytical	Plate theory	% error
(a) $h/L_x = 1/4$					
1	(1,1)	o	0.577	0.579	0.37
2	(1,0), (0,1)	i	0.785	0.789	0.46
3	(1,1)	i	1.111	1.142	2.82
4	(2,1), (1,2)	o	1.186	1.208	1.89
5	(2,0), (0,2)	i	1.571	1.582	0.68
6	(2,2)	o	1.656	1.679	1.41
7	(2,1), (1,2)	i	1.756	1.831	4.28
8	(3,1), (1,3)	o	1.925	1.993	3.54
9	(1,1)	i	2.111	2.118	0.33
10	(2,2)	i	2.221	2.446	10.14
(b) $h/L_x = 1/8$					
1	(1,1)	o	0.167	0.169	0.92
2	(2,1), (1,2)	o	0.386	0.398	3.05
3	(1,0), (0,1)	i	0.393	0.394	0.33
4	(1,1)	i	0.555	0.570	2.68
5	(2,2)	o	0.577	0.591	2.36
6	(3,1), (1,3)	o	0.694	0.739	6.43
7	(2,0), (0,2)	i	0.785	0.791	0.71
8	(3,2), (2,3)	o	0.855	0.891	4.22
9	(2,1), (1,2)	i	0.878	0.913	3.95
10	(1,1)	i	1.097	1.097	−0.03
(c) $h/L_x = 1/12$					
1	(1,1)	o	0.077	0.078	0.88
2	(2,1), (1,2)	o	0.184	0.190	3.18
3	(1,0), (0,1)	i	0.262	0.263	0.30
4	(2,2)	o	0.284	0.291	2.32
5	(3,1), (1,3)	o	0.347	0.368	5.94
6	(1,1)	i	0.37	0.377	1.87
7	(3,2), (2,3)	o	0.437	0.454	3.93
8	(2,0), (0,2)	i	0.524	0.527	0.48
9	(4,1), (1,4)	o	0.55	0.598	8.79
10	(3,3)	o	0.577	0.598	3.57
(d) $h/L_x = 1/20$					
1	(1,1)	o	0.028	0.029	3.81
2	(2,1), (1,2)	o	0.069	0.073	6.19
3	(2,2)	o	0.109	0.114	4.45
4	(3,1), (1,3)	o	0.135	0.148	9.44
5	(1,0), (0,1)	i	0.157	0.158	0.43
6	(3,2), (2,3)	o	0.173	0.184	6.46
7	(1,1)	i	0.222	0.226	1.76
8	(4,1), (1,4)	o	0.223	0.248	11.23
9	(3,3)	o	0.234	0.252	7.56
10	(4,2), (2,4)	o	0.258	0.284	10.05

4.4. Clamped-free square and rectangular plates

When two opposite edges of a square or a rectangular plate are clamped and the other two are free, Table 4 lists the first ten frequencies. The first frequency corresponding to the out-of-plane mode of vibration is higher

Table 2
Comparison of the first 10 normalized natural frequencies of a simply supported rectangular plate ($L_x = 2L_y$) computed from the analytical solution with those obtained from the mixed Kth order shear and normal deformable plate theory. Frequencies of the rectangular plate made of a compressible material computed with the plate theory and the analytical solution are also listed.

Number	Mode		Normalized frequencies (incompressible material)			Mode		Normalized frequencies (compressible material)	
	(m,n)	in-plane/ out-of-plane	Analytical	Plate theory $K = 5$	% error	(m,n)	in-plane/ out-of-plane	Plate theory ($K = 5$)	Analytical
(a) $h/L_x = 1/4$									
1	(1,0)	i	0.785	0.789	0.48	(1,0)	i	0.7857	0.7854
2	(1,1)	o	1.186	1.190	0.37	(1,1)	o	1.0725	1.0692
3	(0,1)	i	1.571	1.578	0.42	(2,1)	o	1.5248	1.5158
4	(2,0)	i	1.571	1.582	0.73	(0,1)	i	1.5715	1.5708
5	(2,1)	o	1.656	1.670	0.84	(2,0)	i	1.5763	1.5708
6	(1,1)	i	1.756	1.791	1.98	(1,1)	i	1.7587	1.7562
7	(2,1)	i	2.221	2.364	6.46	(3,1)	o	2.1500	2.1219
8	(3,1)	o	2.283	2.331	2.10	(2,1)	i	2.2343	2.2214
9	(3,0)	i	2.356	2.386	1.27	(3,0)	i	2.3766	2.3562
10	(1,2)	o	2.702	2.737	1.29	(1,2)	o	2.5490	2.5305
(b) $h/L_x = 1/8$									
1	(1,1)	o	0.385	0.388	0.70	(1,1)	o	0.3349	0.3373
2	(1,0)	i	0.393	0.394	0.35	(1,0)	i	0.3927	0.3929
3	(2,1)	o	0.577	0.585	1.34	(2,1)	o	0.5066	0.5131
4	(0,1)	i	0.785	0.789	0.47	(3,1)	o	0.7606	0.7800
5	(2,0)	i	0.785	0.791	0.75	(0,1)	i	0.7854	0.7858
6	(3,1)	o	0.855	0.888	3.83	(2,0)	i	0.7854	0.7888
7	(1,1)	i	0.878	0.894	1.83	(1,1)	i	0.8781	0.8797
8	(1,2)	o	1.051	1.076	2.37	(1,2)	o	0.9425	0.9550
9	(2,1)	i	1.111	1.176	5.88	(2,2)	o	1.0692	1.0823
10	(3,0)	i	1.178	1.192	1.18	(4,1)	o	1.0692	1.1131
Number	Mode		Normalized frequencies (incompressible material)			Mode		Normalized frequencies (compressible material)	
	(m,n)	in-plane/ out-of-plane	Analytical	Plate theory $K = 3$	% error	(m,n)	in-plane/ out-of-plane	Plate theory ($K = 3$)	Analytical
(c) $h/L_x = 1/12$									
1	(1,1)	o	0.184	0.185	0.59	(1,1)	o	0.1594	0.1581
2	(1,0)	i	0.262	0.263	0.31	(2,1)	o	0.2490	0.2455
3	(2,1)	o	0.284	0.287	1.19	(1,0)	i	0.2619	0.2618
4	(3,1)	o	0.437	0.451	3.26	(3,1)	o	0.3913	0.3811
5	(0,1)	i	0.524	0.526	0.31	(1,2)	o	0.4887	0.4822
6	(2,0)	i	0.524	0.527	0.50	(0,1)	i	0.5237	0.5236
7	(1,2)	o	0.55	0.560	1.86	(2,0)	i	0.5248	0.5236
8	(1,1)	i	0.585	0.593	1.29	(2,2)	o	0.5611	0.5544
9	(2,2)	o	0.63	0.639	1.43	(4,1)	o	0.5768	0.5544
10	(4,1)	o	0.63	0.667	5.82	(1,1)	i	0.5859	0.5854
(d) $h/L_x = 1/20$									
1	(1,1)	o	0.069	0.070	1.72	(1,1)	o	0.0601	0.0589
2	(2,1)	o	0.109	0.112	2.68	(2,1)	o	0.0962	0.0931
3	(1,0)	i	0.157	0.158	0.44	(3,1)	o	0.1567	0.1485
4	(3,1)	o	0.173	0.183	5.79	(1,0)	i	0.1571	0.1571
5	(1,2)	o	0.223	0.229	2.80	(1,2)	o	0.1963	0.1913
6	(2,2)	o	0.258	0.265	2.61	(2,2)	o	0.2279	0.2226
7	(4,1)	o	0.258	0.283	9.77	(4,1)	o	0.2399	0.2226
8	(0,1)	i	0.314	0.315	0.44	(3,2)	o	0.2812	0.2735
9	(2,0)	i	0.314	0.316	0.63	(5,1)	o	0.3438	0.313
10	(3,2)	o	0.316	0.325	2.92	(0,1)	i	0.3142	0.3142

Table 3
First ten normalized natural frequencies of incompressible isotropic square and rectangular clamped plates computed with the mixed HOSNDPT

Number	$h/L_x = 1/4$		$h/L_x = 1/8$		$h/L_x = 1/12$		$h/L_x = 1/20$	
	Mode							
	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency
(a) *Square plate*								
1	o	0.867	o	0.315	o	0.146	o	0.058
2	o	1.405	o	0.565	o	0.279	o	0.116
3	o	1.836	o	0.755	o	0.386	o	0.164
4	i	1.967	o	0.889	o	0.464	o	0.204
5	i	2.012	o	0.904	o	0.470	o	0.206
6	o	2.110	i	1.010	o	0.552	o	0.244
7	o	2.137	i	1.023	i	0.609	o	0.312
8	o	2.452	o	1.041	i	0.627	o	0.319
9	i	2.684	o	1.277	o	0.690	o	0.349
10	i	2.842	o	1.285	o	0.692	o	0.353
(b) *Rectangular plate*, $L_x = 2L_y$								
1	o	1.528	o	0.636	o	0.338	o	0.144
2	o	1.890	o	0.793	o	0.426	o	0.183
3	i	2.304	o	1.050	o	0.576	o	0.256
4	o	2.459	i	1.169	o	0.723	o	0.337
5	o	2.818	o	1.251	i	0.765	o	0.359
6	o	3.093	o	1.366	o	0.775	o	0.368
7	i	3.151	o	1.381	o	0.788	o	0.422
8	o	3.158	o	1.556	o	0.899	i	0.452
9	i	3.305	i	1.592	o	1.009	o	0.486
10	o	3.517	i	1.702	i	1.041	o	0.503

for this plate than that for an identical plate with all edges simply supported but lower than that with all edges clamped.

4.5. Remarks

Qian, Batra and Chen [12] used the compatible HOSNDPT and the meshless local Petrov–Galerkin method with basis functions obtained by the moving least squares approximation to compute frequencies of an isotropic homogeneous plate with $\nu = 0.499$. It thus seems that the meshless method can handle the incompressibility constraint better than the finite element method. Qian and Batra [13] employed the compatible HOSNDPT to find the in-plane distribution of the volume fractions of two constituents so as to optimize the first fundamental frequency of a cantilever plate.

Batra [14] has derived a compatible HOSNDPT for a plate made of an incompressible linear elastic material.

5. Conclusions

It is shown that the mixed HOSNDPT of Batra and Vidoli gives frequencies that agree well with those obtained from the analytical solution, and predicts frequencies that are missed by the classical plate theory and the first order shear deformation theory. The plate theory has been used to compute frequencies of square and rectangular plates that are either simply supported or clamped or have two edges clamped and the other two edges traction free.

Table 4
First ten normalized natural frequencies of incompressible isotropic square and rectangular clamped–free plates computed with the mixed HOSNDPT

Number	$h/L_x = 1/4$		$h/L_x = 1/8$		$h/L_x = 1/12$		$h/L_x = 1/20$	
	Mode							
	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency	in-plane/ out-of-plane	Frequency
(a) *Square plate*								
1	o	0.580	o	0.199	o	0.093	o	0.037
2	o	0.615	o	0.216	o	0.102	o	0.040
3	i	0.747	o	0.327	o	0.159	o	0.063
4	o	0.959	i	0.373	o	0.238	o	0.099
5	o	1.194	o	0.474	i	0.247	o	0.104
6	o	1.253	o	0.494	o	0.249	o	0.115
7	i	1.423	o	0.582	o	0.288	o	0.126
8	i	1.504	o	0.593	o	0.304	i	0.148
9	o	1.540	i	0.710	o	0.414	o	0.173
10	o	1.628	i	0.747	o	0.432	o	0.190
(b) *Rectangular plate*								
1	o	1.385	o	0.581	o	0.308	o	0.130
2	o	1.403	o	0.590	o	0.314	o	0.133
3	i	1.544	o	0.667	o	0.353	o	0.151
4	o	1.602	i	0.770	o	0.444	o	0.190
5	o	2.083	o	0.831	i	0.511	o	0.260
6	i	2.116	i	1.063	o	0.602	i	0.306
7	o	2.678	o	1.127	o	0.693	o	0.324
8	o	2.722	o	1.199	o	0.700	o	0.327
9	o	2.745	o	1.213	i	0.704	o	0.347
10	i	2.925	o	1.292	o	0.744	o	0.362

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